C^0-COERCIVENESS OF MOSER'S PROBLEM AND SMOOTHING AREA PRESERVING HOMEOMORPHISMS

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Abstract. In this paper, we establish the C^0-coerciveness of Moser's problem of mapping one smooth volume form to another in terms of the weak topology of measures associated to the volume forms. The proof relies on our analysis of Dacorogna-Moser's solution to Moser's problem of mapping one volume form to the other with the same total mass. As an application, we give a proof of smoothing result of area preserving homeomorphisms and its parametric version in two dimension, (or more generally in any dimension in which the smoothing theorem of homeomorphisms is possible, e.g., in dimension 3 but not necessarily in dimension 4). This in turn results in coincidence of the area-preserving homeomorphism group and the symplectic homeomorphism group in two dimension.

Contents

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1. Introduction and the main theorems

Consider a symplectic manifold (X,ω) and denote by $Diff(X)$ the group of smooth diffeomorphisms of X. Eliashberg's celebrated C^0 rigidity theorem [E], [G1] in symplectic geometry states that the subgroup $Symp(X,\omega)$ of $Diff(X)$ consisting of symplectic diffeomorphisms, i.e., those η satisfying $\eta^*\omega = \omega$ is C^0 closed in $Diff(X)$. More precisely, we equip the group $Homeo(X)$ of homeomorphisms with the metric defined as

$$\bar{d}(h,k) = \max_{x\in X}(d(h(x),k(x)) + d(h^{-1}(x),k^{-1}(x)))$$

where d is a distance of any given Riemannian metric. With this metric, $Homeo(X)$ becomes a topological group which is a complete metric space. We consider the induced topology on $Diff(X) \subset Homeo(X)$. Eliashberg's rigidity theorem then can be phrased as $Symp(X,\omega)$ is a *closed* topological subgroup of $Diff(X)$ with respect to this induced topology. Motivated by this rigidity theorem, we defined

$$Sympeo(X,\omega) := \overline{Symp}(X,\omega)$$

where $\overline{Symp}(X,\omega)$ is the closure of $Symp(X,\omega)$ in $Homeo(X)$, and called this group the group of *symplectic homeomorphisms* [OM]. With this definition, the rigidity theorem can be succinctly written as

$$Sympeo(X,\omega) \cap Diff(X) = Symp(X,\omega).$$

Then in the same paper [OM], we introduced the notion of *Hamiltonian homeomorphisms* and denote the set thereof by $Hameo(X,\omega)$. This is the C^0 counterpart of the group $Ham(X,\omega)$ of Hamiltonian diffeomorphisms. We also proved that $Hameo(X,\omega)$ forms a path-connected *normal subgroup* of $Sympeo_0(X,\omega)$, and conjectured that $Hameo(X,\omega)$ is a *proper* subgroup of $Sympeo_0(X,\omega)$. We refer readers to [OM] for further discussions on the structure of the Hamiltonian homeomorphism group.

In two dimensional compact surface (Σ,Ω) with an area form Ω, we denote by $Homeo^{\Omega}(\Sigma)$ the group of Ω-area preserving homeomorphisms on Σ. It easily follows from the definition that $Sympeo(\Sigma,\Omega)$ is the subgroup of $Homeo^{\Omega}(\Sigma)$ that consists of area preserving homeomorphisms approximable by area preserving (smooth) diffeomorphisms.

The main motivation of the present paper is to prove the following result conjectured in [OM].

Theorem I. *For a two dimensional surface (Σ,ω), we also write $\omega = \Omega$ as an area form. Then we have*

$$Sympeo(\Sigma,\omega) = Homeo^{\Omega}(\Sigma), \quad Sympeo_0(\Sigma,\omega) = Homeo_0^{\Omega}(\Sigma).$$

Here we denote by G_0 the identity component of any topological group G.

Theorem I and normality of $Hameo(D^2,\partial D^2)$ in $Sympeo(D^2,\partial D^2)$ and path-connectedness of $Hameo(S^2,\Omega)$ proven in [OM] are the bases on the conjecture on the structure of $Homeo^{\Omega}(D^2,\partial D^2)$ made in [OM], [OF], which reads that $Homeo^{\Omega}(D^2,\partial D^2)$ *is not a simple group.*

In more concrete terms, this theorem can be rephrased as the following smoothing result of area preserving homeomorphisms which is one belonging solely to the realm of area preserving dynamical system. This smoothing result seems to have been a

folklore among the experts in the area but we could not locate any proper reference containing its proof in the literature.

Theorem I$'$. *Let Σ be a compact surface without boundary and Ω be an area form. Denote by μ_Ω the Borel measure induced by the integration of Ω. Then,*

(1) *any area preserving homeomorphism h can be C^0 approximated by an area preserving diffeomorphism*
(2) *any isotopy $h = \{h_t\}_{0\leq t\leq 1}$ with $h_0 = id$ of area preserving homeomorphisms can be C^0 approximated by a smooth isotopy of area preserving diffeomorphisms.*

As our proof will show, Theorem I$'$ holds for any Borel measure induced by a volume form (or by a volume density if not orientable) on general compact manifolds in general dimension, *as long as approximation of any homeomorphism on a manifold X by a diffeomorphism is possible*, for example in dimension 2 and 3 [Mu2] but possibly not in dimension 4 [D]. It seems to be an interesting open question to ask whether the measure preserving property helps one to approximate a homeomorphism by a diffeomorphism and so to prove Theorem I$'$ in complete generality in high dimensions.

To highlight the main point of the present paper, we outline our proof of (1) here. Denote by $M[\Sigma,\Omega] = Homeo^\Omega(\Sigma)$ the topological group of measure preserving homeomorphisms on (Σ,μ_Ω) equipped with the topology induced by the metric $\bar{d}$ defined above. We call this topology the C^0 topology of $Homeo^\Omega(\Sigma)$. We will also denote by d_{C^0} the usual C^0 metric given by

$$d_{C^0}(h,k) = \max_{x\in X} d(h(x),k(x)).$$

Let $h \in M[\Sigma,\Omega]$ and $\varepsilon > 0$ be given. By the well-known smoothing theorem (see the proof of Theorem 6.3 [Mu2], for example) for $\dim X = 2$, we can choose a diffeomorphism ψ_1 such that

$$\bar{d}(h,\psi_1) \leq \frac{\varepsilon}{3}. \tag{1.1}$$

This diffeomorphism ψ_1 however may *not* necessarily be area preserving. We therefore modify ψ_1 into an area preserving diffeomorphism by a C^0 small perturbation.

Here we would like to emphasize that *the two forms $\psi_1^*\Omega$ and Ω are not necessarily C^0 close.* More precisely, we have

$$\psi_1^*\Omega = f\Omega, \quad f > 0$$

where $f = \det d\psi_1$ with $d\psi_1$ being the derivative of ψ_1. Since we do not have any control on $d\psi_1$ in the C^0 convergence, the modulus $|f-1|$ is not necessarily small. We denote

$$|g| = \max_{x\in\Sigma} |g(x)|$$

for a function g in general. However it is not difficult to see that (1.1) also implies that the measures associated to Ω and $\psi_1^*\Omega$ can be made arbitrarily close in the *weak topology* of measures if one chooses ε sufficiently small. (See Proposition 2.1.)

It is well-known that the set $\mathcal{M}(X)$ of finite measures on a compact metric space X is a metric space such that the subset $\mathcal{M}^m(X)$ of measures whose total mass is less than equal to $m \in \mathbb{R}_+$ is compact. (See [G2] for example.) We denote by $d_{\mathcal{M}}$ a corresponding metric on $\mathcal{M}(X)$. Now we will derive the proof of Theorem I$'$ from the following theorem concerning coerciveness of the C^0 distance with respect

to the weak topology of measures. *This theorem holds in arbitrary dimension.* We assume X is orientable for the simplicity. Non-orientable case will be the same if we replace the volume form by the density. We denote by μ_σ the measure induced by the volume form σ in general.

The main result of the present paper is then the following C^0-coerciveness of such diffeomorphisms ψ_2 in terms of the distance $d_{\mathcal{M}}(\mu_{\psi_1^*\Omega}, \mu_\Omega)$ or in terms of the weak topology of measures.

Theorem II. *Let σ and τ be two volume forms $\sigma = f\tau$ on X with f satisfying $f > 0$. Let $\lambda > 0$ be the constant*

$$\lambda = \int_X \sigma \Big/ \int_X \tau.$$

Then there exists a diffeomorphism $\psi_2 : X \to X$ such that

$$\psi_2^* \sigma = \lambda \tau.$$

Furthermore, we have

$$\bar{d}(\psi_2, id) \to 0 \quad \text{as } d_{\mathcal{M}}(\mu_\sigma, \mu_\tau) \to 0. \tag{1.2}$$

Moreover its parametric version also holds : For any isotopy of forms $t \in [0,1] \mapsto f_t\tau$ where $t \mapsto \mu_{(f_t\tau)}$ defines a continuous path in $\mathcal{M}(X)$, there exists an isotopy $t \in [0,1] \to \psi_{2,t}$ of diffeomorphisms satisfying $\psi_{2,t}^*\sigma = \lambda_t \tau$ that is continuous in the compact open topology.

In fact, our proof of the parameterized version of Theorem II provides canonical local slices of the action of $Homeo(X)$ on $\mathcal{M}(X)$

$$\Psi_{\mu_0} : U_{\mu_0} \cap \mathcal{M}(X;\Omega) \to Homeo(X)$$

around $\mu_0 = \mu_{g\Omega}$ with g continuous, where $U_{\mu_0} \subset \mathcal{M}(X)$ is an open neighborhood of in $\mathcal{M}(X)$, and $\mathcal{M}(X;\Omega)$ is the space of measures that are absolutely continuous with respect to μ_Ω. We will elaborate this generalization elsewhere.

Once we have Theorem II, we apply the theorem to the forms

$$\sigma = \Omega, \quad \tau = (\psi_1^{-1})^*\Omega, \quad \text{with } \lambda = 1$$

and construct ψ_2 such that

$$(\psi_2)^*\Omega = (\psi_1^{-1})^*\Omega \quad \text{and } \bar{d}(\psi_2, id) \le \frac{\varepsilon}{3}$$

by letting $d_{\mathcal{M}}(\mu_\Omega, \mu_{(\psi_1^{-1})^*\Omega})$ as small as we want. The last can be achieved if we choose ψ_1 sufficiently C^0 close to the area preserving homeomorphism h. Then we prove that the composition $\phi := \psi_2 \circ \psi_1 : X \to X$ is an area preserving diffeomorphism with the estimate

$$\bar{d}(\phi, id) \le \varepsilon$$

for any given $\varepsilon > 0$. A simple examination of the proof will also give rise to the proof of its parametric version. This will then finish the proof of Theorem II and so Theorem I$'$.

Theorem II *without (1.2)* is a result proven by Moser [Mo]. And the $C^{k+1,\alpha}$ estimate for $k \ge 0$, $0 < \alpha < 1$ that is the Hölder analog to (1.2) was also proven by Dacorogna and Moser [DM]. The main point of Theorem II is the C^0-coerciveness with respect to the weak topology of measures which is a crucial ingredient in our proof of the smoothing theorem, Theorem I$'$. We prove this coerciveness by

analyzing the C^0-behavior of Dacorogna-Moser's solution on the cube obtained by their 'elementary approach' with respect to the weak topology of measures.

For this purpose, we first have to turn Dacorogna and Moser's original *one-dimensional* scheme into an *n-dimensional* scheme which optimally reflects its n-dimensional measure theoretic behavior, and to use sufficiently small cubes whose size depends only on the given reference volume form on X. (See section 3, especially Remark 3.1, and the proof of Proposition 6.1.) Furthermore we like to mention that, interestingly enough, *open mapping theorem* plays an essential role in our derivation of C^0-coercive estimates of Dacorogna-Moser's solution with respect to the weak topology of measures. (See section 5.)

Now we mention some related results in the literature. In their seminal paper, among other things, Oxtoby and Ulam [OU] proved an approximation of measure preserving homeomorphisms by *almost everywhere* differentiable measure preserving homeomorphisms. Our proof relies on a smoothing result of general homeomorphisms for $n = 2$. This result can be extracted from [Mu1], [Mu2] and the references therein, for example. We refer readers to the proof of Theorem 6.3 [Mu2] in particular. The result Theorem I$'$ itself seems to have been a folklore among the experts. However we have not been able to locate a proper reference containing its proof (or its statement) in the literature. The main theorem, Theorem II, has its own separate interest in its possible relation to the study of generalized flows of incompressible perfect fluids and to the problem of optimal transport (See [Br], [Sh], [Vi] for example.)

Organization of the contents is in order. Section 2 summarizes the basic facts on the weak topology of measures relevant to the proofs of Theorem I$'$ and II. Section 3 recalls and enhances Moser's reduction procedure [Mo] of the problem to one on the cube establishing continuity of the procedure in the weak topology of measures. Section 4 reviews Dacorogna-Moser's elementary approach closely and provides a reformulation of their scheme so that we can analyze its dependence on the weak topology of measures. After then, section 5-7 contain the proof of Theorem II. The proof of Theorem I$'$ will be finished in section 8. Finally in section 9, we prove the key a priori estimates for the study of C^0-coercive estimates of Dacorogona-Moser's solution. This last section contains the most technical estimates of the paper, whose validity, however, is motivated by Taylor's remainder theorem.

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Notations.

(1) $Q = Q^n = [0, 1]^n$, the unit cube in $\mathbb{R}^n$
(2) $Q^n(K) = [0, K]^n$, the cube with its size length $K > 0$
(3) For a positive number η with $0 < \eta < 1$, we denote

$$\begin{aligned} Q^n(1+\eta) &= \{x \in \mathbb{R}^n \mid -\eta \leq x_j \leq 1+\eta,\ j = 1, \cdots, n\} \\ Q^n(1-\eta) &= \{x \in \mathbb{R}^n \mid \eta \leq x_j \leq 1-\eta,\ j = 1, \cdots, n\} \end{aligned}$$

(4) For a vector $a \in Q^n$ and $b \in [-\eta, \eta]^n$, we denote

$$x^{n-1}_{a;j} = (x^{j-1}, a_j, \widetilde{x}_j) \quad \text{for } j = 1, \cdots, n$$

where we denote $\widetilde{x}_j := (x^{j+1}, \cdots, x^n)$.
(5) $R^n_{au;j}$: see (4.30).
(6) $Q^n_{a;j}$: see (4.29).
(7) $R^{n-1}_{au;j,k}$: see (9.7).
(8) $Q^n_{a;j,k}$: see (9.2).
(9) $C^{\vec{1}}(Q^n, \mathbb{R}^n)$: see Definition 5.1.

2. Weak topology of $\mathcal{M}(X)$

In this section, we briefly review the *weak topology* of the space of finite measures on a compact metric space X following the exposition from section $3\frac{1}{2}.9$ [G2].

Definition 2.1 (Weak topology). A sequence of finite measures μ_i is said to converge to μ if $\mu_i(f) - \mu(f)$ for every bounded, nonnegative, continuous function f on X, where $\mu(f)$ stands for $\int_X f d\mu$. We denote by $\mathcal{M}(X)$ the set of finite measures equipped with this topology.

It turns out the weak topology is induced by a metric. One such metric can be defined by

$$\mathrm{Lid}_b(\mu, \mu') := \sup_f |\mu(f) - \mu'(f)| \tag{2.1}$$

for $b > 0$, where f runs over all *1-Lipschitz* functions $f : X \to [0, b]$. These define true metrics on $\mathcal{M}(X)$ and they are mutually bi-Lipschitz equivalent. The metrics are also complete and if X is compact, then the subset of $\mathcal{M}(X)$

$$\mathcal{M}_m = \{\mu \in \mathcal{M}(X) \mid \mu(X) \leq m\}$$

is compact for each fixed $m \in \mathbb{R}_+$. We denote

$$d_{\mathcal{M}} = \mathrm{Lid}_1 .$$

There is a natural map

$$Homeo(X) \times \mathcal{M}(X) \to \mathcal{M}(X); (h, \mu) \mapsto h_* \mu \tag{2.2}$$

which is continuous (see Proposition 1.5, [F] for example).

Next we consider the Borel measures induced by volume forms. Let Ω be a volume form on a compact manifold X satisfying $|\Omega| := \int_X \Omega < \infty$, and denote by μ_Ω the measure induced by integrating the form Ω. Denoting by $\Omega^n(X)$ the space of volume forms, there is a natural action of $Diff(X)$

$$Diff(X) \times \Omega^n(X) \to \Omega^n(X); \quad (\psi, \Omega) \mapsto \psi^* \Omega \tag{2.3}$$

which is continuous in C^∞ topology. It also induces a map

$$Diff(X) \times \Omega^n(X) \to \mathcal{M}(X); \quad (\psi, \Omega) \mapsto \psi_*(\mu_\Omega) = \mu_{(\psi^*\Omega)}.$$

The following proposition will play an essential role in our proof.

Proposition 2.1. *Let $\psi \in Diff(X)$ and $h \in Homeo^\Omega(X)$ and $\psi \to h$ in C^0 topology. Then we have*

$$\mu_{(\psi^*\Omega)} \to \mu_\Omega \quad \text{in } \mathcal{M}(X).$$

The convergence is uniform over any given compact family of hs.

Proof. It follows that $\mu_{(\psi^*\Omega)} = \psi_*(\mu_\Omega)$. Since $\psi \to h$ in C^0, continuity of (2.2) implies

$$\psi_*(\mu_\Omega) \to h_*(\mu_\Omega)$$

in $\mathcal{M}(X)$. On the other hand we have $h_*(\mu_\Omega) = \mu_\Omega$ by the hypothesis $h \in Homeo^\Omega(X)$. This finishes the proof of the first statement. The second statement is an immediate consequence of the compactness assumption of the family. $\square$

Now denote

$$C^0(X, \mathbb{R}_+) = \{f \in C^0(X, \mathbb{R}) \mid f > 0\}$$

and consider a volume form Ω. Ω induces a natural embedding

$$\iota_\Omega : C^0(X, \mathbb{R}_+) \hookrightarrow \mathcal{M}(X)$$

defined by

$$\iota_\Omega(f) := \mu_{(f\Omega)}.$$

This is a Lipschitz map which satisfies

$$d_{\mathcal{M}}(\mu_{(f\Omega)}, \mu_{(f'\Omega)}) \leq |\Omega| \cdot |f - f'|. \tag{2.4}$$

3. Reduction of Theorem II to the cube

In this section, we reduce the proof of Theorem II to the case of the cube $Q = [0,1]^n \subset \mathbb{R}$. This reduction will be based on a refinement of Lemma 1 [Mo], Proposition 3.1 below. The main refinements lie in the statements (1) and (3) thereof.

Let $\{U_0, \cdots, U_m\}$ be an open covering of X each element of which can be mapped one to one onto the unit cube $Q = (0,1)^n$.

Proposition 3.1 (Compare with Lemma 1 [Mo]). *Let X be a compact manifold without boundary and let Ω and $f\Omega$ be a volume form and f a positive function satisfying $\int \Omega = \int f\Omega$. Then there exists decomposition of $g = f - 1$*

$$g = \sum_{j=0}^{m} g_j$$

where g_j has support in U_j, and satisfies the following properties :

1. *For all $k = 0, \cdots, m$, we have*
$$f_k := 1 + \sum_{i=0}^{k} g_i > 0 \tag{3.1}$$
and in particular $f_k\Omega$ defines a natural measure $\mu_{(f_k\Omega)}$ by integrating the form $f_k\Omega$.
2. *For all $k = 0, \cdots, m$,*
$$\int g_k\Omega = 0 \quad \text{or equivalently} \quad \int \Omega = \int f_k\Omega. \tag{3.2}$$
3. *We have*
$$d_{\mathcal{M}}(\mu_{(f_k\Omega)}, \mu_{(f\Omega)}) \leq C_1 \tag{3.3}$$
where $C_1 = C_1(d_{\mathcal{M}}(\mu_\Omega, \mu_{(f\Omega)})$ is a constant depending on $d_{\mathcal{M}}(\mu_\Omega, \mu_{(f\Omega)})$ and the covering only and satisfying $C_1 \to 0$ as $d_{\mathcal{M}}(\mu_\Omega, \mu_{(f\Omega)}) \to 0$.
4. *If $g \in C^k$, so is $g_j \in C^k$, $k \geq 0$.*

Proof. Following the proof of Lemma 1 [Mo], we choose a partition of unity $\phi_j \geq 0$ subordinate to the covering $U_0, \cdots, U_m$. We order the elements U_j so that for every $k = 1, \cdots, m$ the patch U_k intersects $\cup_{j<k} U_j$. We denote by $\rho(k)$ any integer with $\rho(k) < k$ such that $U_k \cap U_{\rho(k)} \neq \emptyset$. Then define the matrix $\alpha = (\alpha_{jk})$ with $0 \leq j \leq m$ and $1 \leq k \leq m$ by

$$\alpha_{jk} = \begin{cases} 1 & \text{for } j = k \\ -1 & \text{for } j = \rho(k), \\ 0 & \text{otherwise.} \end{cases}$$

This matrix satisfies $\sum_{j=0}^m \alpha_{jk} = 0$.

We now fix functions η_k, $k = 1, \cdots, m$ such that

$$\int \eta_k \Omega = 1. \tag{3.4}$$

We can choose them so that

$$|\eta_k| \leq C_2$$

where C_2 depends only on the covering and Ω. We will represent g_j in the form

$$g_j = g\phi_j - \sum_{k=1}^m \lambda_k \alpha_{jk} \eta_k.$$

Then Moser [Mo] showed that g_j is add up to g and has support in U_j. To prove (3.2), we consider the linear equation

$$\sum_{k=1}^m \lambda_k \alpha_{jk} = \int_X (f-1)\phi_j \Omega \tag{3.5}$$

for $j = 0, \cdots, m$, which has m unknowns and $m+1$ equations. However, on account of (3.4) and the equation

$$\int (f-1)\Omega = \sum_{j=0}^m \int_X (f-1)\phi_j \Omega = 0$$

the first equation ($j = 0$) of (3.5) is redundant. Therefore the solution space of (3.5) is a nonempty affine subspace of $\mathbb{R}^m$. So far our proof has been a duplication of Moser's [Mo].

The new statements in this proposition that were not considered in [Mo] or [DM] are (3.1) and (3.3). To establish these statements, we need to analyze the solution space of (3.5) more closely than [Mo] or [DM] do in terms of the weak topology of measures. First we note that we have

$$f_j = 1 + \sum_{i=0}^j \left(\phi_i (f-1) - \sum_{k=1}^m \lambda_k \alpha_{ik} \eta_k \right).$$

We set $f_{-1} = 1$ and note $f_m = f$. Thanks to (2.4), to prove (3.1) and (3.3), it will be enough to make the norms $|\lambda_k|$ all sufficiently small. To be more precise, we rewrite (3.5) into

$$\sum_{k=1}^m \lambda_k \alpha_{jk} = \int_X \phi_j d\mu_{(f\Omega)} - \int_X \phi_j d\mu_\Omega$$

for $j = 0, \cdots, m$. Recalling the definition of $d_{\mathcal{M}} = \mathrm{Lid}_1$, note that the right hand side is bounded by

$$\Big| \int_X \phi_j d\mu_{(f\Omega)} - \int_X \phi_j d\mu_\Omega \Big| \leq d_{\mathcal{M}}(\mu_{(f\Omega)}, \mu_\Omega)$$

since ϕ_j is a function satisfying $0 \leq \phi_j \leq 1$. We like to alert the readers that *the distance in the right side of this inequality is in terms of the weak topology of measure.*

A simple linear algebra then concludes that there exist solutions λ_k of (3.5) such that

$$|\lambda_k| \leq C_3, \quad k = 1, \cdots, m \tag{3.6}$$

where $C_3 = C_3(d_{\mathcal{M}}(\mu_{(f\Omega)}, \mu_\Omega))$ is a constant depending only on $d_{\mathcal{M}}(\mu_{(f\Omega)}, \mu_\Omega)$ such that $C_3 \to 0$ as $d_{\mathcal{M}}(\mu_{(f\Omega)}, \mu_\Omega) \to 0$: Note that the solution space of (3.5) is a nonempty affine subspace of $\mathbb{R}^m$, whose distance from the origin converges to zero as $d_{\mathcal{M}}(\mu_{(f\Omega)}, \mu_\Omega) \to 0$. To obtain such a solution $(\lambda_1, \cdots, \lambda_m)$ satisfying (3.6), one may take the point nearest to the origin among the points in the affine space.

Now to prove (3.1), we consider the convex combinations of f and 1

$$f'_j := 1 + \left(\sum_{i=0}^{j} \phi_i\right)(f - 1) = \left(\sum_{i=0}^{j} \phi_i\right) f + \left(1 - \left(\sum_{i=0}^{j} \phi_i\right)\right) 1$$

for $j = 0, \cdots, m$ and denote

$$\begin{aligned} m_0 &= \min_j \{\min f'_j \mid j = 0, \cdots, m\} \\ M_0 &= \max_j \{\max f'_j \mid j = 0, \cdots, m\} \end{aligned}$$

Note that m_0, M_0 depends only on f and satisfies

$$m_0 \geq \min\{\min f, 1\}, \quad M_0 \leq \max\{\max f, 1\} \tag{3.7}$$

Recalling that $C_3 \to 0$ as $d_{\mathcal{M}}(\mu_{(f\Omega)}, \mu_\Omega) \to 0$, we can choose $d_{\mathcal{M}}(\mu_{f\Omega}, \mu_\Omega)$ so small that we have

$$C_3 < \frac{\min\{\min f, 1\}}{m(m+1)C_2}.$$

Then we derive

$$\max \sum_{k=1}^{m} |\lambda_k||\eta_k| \leq mC_2C_3 < \frac{\min\{\min f, 1\}}{m+1} \tag{3.8}$$

from (3.6). Therefore we have

$$\begin{aligned} f_j &= f'_j - \sum_{i=0}^{j} \sum_{k=1}^{m} \lambda_k \alpha_{ik} \eta_k \geq f'_j - \sum_{i=0}^{j} \sum_{k=1}^{m} |\lambda_k||\eta_k| \\ &\geq f'_j - (m+1) \sum_{k=1}^{m} |\lambda_k||\eta_k| > f'_j - \min\{\min f, 1\} \\ &\geq M_0 - \min\{\min f, 1\} \geq 0 \end{aligned}$$

which proves (3.1).

Finally we consider $d_{\mathcal{M}}(\mu_{(f_k\Omega)}, \mu_{(f\Omega)})$ for the proof of (3.3). Since f'_k is a convex combination of f and 1, we have

$$d_{\mathcal{M}}(\mu_{(f'_k\Omega)}, \mu_{(f\Omega)}) \leq d_{\mathcal{M}}(\mu_\Omega, \mu_{(f\Omega)}) \tag{3.9}$$

for any $k = 1, \cdots, m$. We have

$$d_{\mathcal{M}}(\mu_{(f'_k\Omega)}, \mu_{(f_k\Omega)}) \leq |\Omega| \cdot |f'_k - f_k|$$

from (2.4) and

$$|f'_k - f_k| = \Big| \sum_{k=1}^m \lambda_k \alpha_{jk} \eta_k \Big| \leq \sum_{k=1}^m |\lambda_k| \cdot |\eta_k|.$$

On the other hand, we can make $\max(\sum_{k=1}^m |\lambda_k||\eta_k|)$ as small as we want by choosing λ_k small which in turn can be achieved by (3.6) if we make $d_{\mathcal{M}}(\mu_{(f\Omega)}, \mu_\Omega)$ sufficiently small. This implies that we can also make $d_{\mathcal{M}}(\mu_{f'_k\Omega}, \mu_{f_k\Omega})$ as small as we want if we make $d_{\mathcal{M}}(\mu_{(f\Omega)}, \mu_\Omega)$ sufficiently small. We note the triangle inequality

$$d_{\mathcal{M}}(\mu_{(f_k\Omega)}, \mu_{(f\Omega)}) \leq d_{\mathcal{M}}(\mu_{(f_k\Omega)}, \mu_{(f'_k\Omega)}) + d_{\mathcal{M}}(\mu_{(f'_k\Omega)}, \mu_{(f\Omega)}).$$

The last statement of the proposition is obvious from the construction of g_j's. This finishes the proof. □

With Proposition 3.1 in our hand, Theorem II will be derived from the following proposition. *Except the coerciveness (3.12), this is precisely Lemma 2* [Mo] *or Proposition 8* [DM]. However our diffeomorphism may not necessarily the same as the one constructed in [DM]. In fact, our construction will provide *continuous local slices* under the action

$$Homeo(X) \times \mathcal{M}(X) \to \mathcal{M}(X)$$

over a certain dense subset of $\mathcal{M}(X)$. We will elaborate this generalization elsewhere.

Theorem 3.2. *Let Q be the square $[0,1]^n$. Consider two volume forms*

$$\tau = f(x)dx, \quad \sigma = g(x)dx$$

where g, f are positive continuous functions for which $g - f$ has support in Int Q. Denote by m_f, m_g the associated measures. If

$$\int_Q f\, dx = \int_Q g\, dx \tag{3.10}$$

then there exists a diffeomorphisms $\psi : Q \to Q$ such that

$$g(\psi(x)) \det \nabla\psi(x) = f(x) \tag{3.11}$$

such that $\psi(x) = x$ near the boundary of Q. Furthermore ψ satisfies the following additional properties :

(1) *We can make $\bar{d}(\psi, id)$ as small as we want by letting $d_{\mathcal{M}}(m_f, m_g) \to 0$, or*

$$\bar{d}(\psi, id) \to 0 \quad \text{as } d_{\mathcal{M}}(m_f, m_g) \to 0. \tag{3.12}$$

And the parametric version in the sense as stated in Theorem II also holds.

(2) *Let $\operatorname{supp}(\psi) = \overline{\{x \in Q \mid \psi(x) \neq x\}}$. Let $R^n \subset (0,1)^n$ be any closed cube such that*

$$R^n \subset \operatorname{supp}(f - g). \tag{3.13}$$

Then we have

$$\operatorname{supp}(\psi_2) \subset R^n \tag{3.14}$$

Remark 3.1. (1) Obviously, we can further decompose the cube $[0,1]^n$ or use cubes of the smaller size in Proposition 3.1, and get the same kind of statement for the smaller cubes. Later in our estimates, we will need to choose a cube $Q^n(K)$ of its side length $K > 0$ such that K is sufficiently small and depends essentially on the given fixed g. In fact, we can choose K of the form $K = 2^{-N_0}$ with

$$2^{-N_0} < \frac{1}{8C_4(1+L_g)}$$

where $C_4 = \max\{8\max g, 4\}$ and L_g is the modulus of continuity of g. See the paragraph around (6.15) for more discussion on this. However to make our exposition better comparable to that of [DM], we will carry our discussion on the unit cube and just indicate the needed changes in the paragraph around (6.15).

(2) We also note that the above reduction procedure to the cube shows that the distance $d_{\mathcal{M}}(m_f, m_g)$ for the measures m_f, m_g on $Q^n(K)$ converges to zero uniformly as $d_{\mathcal{M}}(\mu_\tau, \mu_\sigma) \to 0$ for the originally given measures μ_τ, μ_σ on X.

(3) The inequality (3.7) shows that the above reduction procedure essentially does not decrease the lower bound $\min g$ and not increase the upper bound $\max g$ on the cube from that of the originally given g on X. *We would like to warn the readers that f in the proof of Proposition 3.1 plays the role of g in Theorem 3.2 and henceforth.*

The next three sections will be occupied by the proof of statement (1) of this theorem.

4. Scheme of construction on the cube

In this section, we first recall Moser's or Dacorogna and Moser's 'elementary approach' from [Mo], [DM] to solving (3.11). After then we reformulate their scheme into an n-dimensional scheme so that we can study its relevance to the weak topology of n-dimensional measures. Their inductive one-dimensional approach as it is does not manifest the relationship of their solutions with the weak topology of measures. We also briefly mention the parametric version of Dacorogna-Moser's approach which is used in the proof of $Sympeo_0(\Sigma,\omega) = Homeo_0^\Omega(\Sigma)$ in Theorem I.

We denote $Q = Q^n = [0,1]^n$ and by Q^s the s dimensional cube for $1 \le s \le n$.

4.1. **Review of Dacorogna and Moser's elementary approach.** According to Dacorogna and Moser [DM], under the assumption as in Theorem 3.2 on f and g, the map $\psi : Q \to Q$ is constructed as the successive composition

$$\psi = \varphi_n \circ \varphi_{n-1} \circ \cdots \circ \varphi_2 \circ \varphi_1$$

by defining $g_n = g$ and for $s = 2, 3, \cdots, n$ and requiring

$$\int_E g_{s-1}(x)\,dx = \int_{\varphi_s(E)} g_s(x)\,dx \tag{4.1}$$

for every open set $E \subset Q^n$ and

$$\int_0^1 g_1(x_1, x')\,dx_1 = \int_0^1 f(x_1, x')\,dx_1. \tag{4.2}$$

And $\varphi_1 : Q \to Q$ will then have the form

$$\varphi_1 : (x_1, x_2, \cdots, x_n) \to (v(x), x_2, \cdots, x_n)$$

where $v : Q \to Q$ is uniquely determined by the requirement

$$\int_0^a f(x_1, x') \, dx_1 = \int_0^{v(a,x')} g_1(x_1, x') \, dx_1 \tag{4.3}$$

for every $x' = (x_2, \cdots, x_n) \in Q^{n-1}$. Since $g_1 > 0$ (4.3) uniquely determines $v(x)$ with v monotone in x_1, $v = 0$ for $x_1 = 0$ and $v = x_1$ for x' near ∂Q^{n-1}. Finally (4.2) makes $v(1, x') = 1$ for all $x' \in Q^{n-1}$. It follows that $g_s \in C^k$, $k \geq 1$, (4.1) is equivalent to

$$g_{s-1}(x) = g_s(\varphi_s(x)) \det \nabla\varphi_s(x). \tag{4.4}$$

Then they construct $\varphi_n, \cdots, \varphi_2$ (and $g_{n-1}, \cdots, g_1$) inductively in such a way that

$$\int_{Q^s} g_s(x^s, x') \, dx^s = \int_{Q^s} f(x^s, x') \, dx^s \tag{4.5}$$

where $x^s = (x_1, \cdots, x_s)$ and $x' = (x_{s+1}, \cdots, x_n)$. Assuming that $\varphi_n, \cdots, \varphi_{s+1}$ are already constructed so that (4.1) and (4.5) hold and that they agree with the identity near the boundary, the map $\varphi_s : Q \to Q$ is constructed as the homeomorphism of the form

$$\varphi_s(x_1, \cdots, x_n) = (x^{s-1}, v(x), x') = (x_1, \cdots, x_{s-1}, v(x), x_{s+1}, \cdots, x_n) \tag{4.6}$$

with

$$v(x) = x_s + \zeta(x^{s-1}) u(x_s, x'). \tag{4.7}$$

Here ζ is a cut-off function with supp $\zeta \subset \text{Int } Q^{s-1}$ and satisfying

$$\begin{cases} 0 \leq \zeta \leq 1 + \varepsilon \quad \text{in } Q^{s-1} \\ \int_{Q^{s-1}} \zeta(x^{s-1}) \, dx^{s-1} = 1 \\ \int_{Q^{s-1}} |\zeta(x^{s-1}) - 1| \, dx^{s-1} < \varepsilon \end{cases} \tag{4.8}$$

where $\varepsilon = \varepsilon(g_s, f) > 0$ is chosen so that

$$\varepsilon \max g_s < \min g_s, \, \frac{1}{2} \min f. \tag{4.9}$$

And $u : [0,1] \to [0,1]$ is a smooth function with

$$u \equiv 0 \quad \text{near } \{0, 1\}.$$

Note that in this construction the variable x' enters only as a parameter and does not play any role in finding φ_s. Therefore we drop x' in our discussion below writing $u(x_s) = u(x_s; x')$ as in [DM]. We refer readers to (4) and (5) [DM] for more details. It follows that φ_s is C^0 close to identity if and only if the one variable function $u : [0,1] \to [0,1]$ is C^0 close to the zero function. Furthermore it becomes a differentiable homeomorphism if and only if u is differentiable and satisfies

$$\frac{\partial v}{\partial x_s} = 1 + \zeta(x^{s-1}) \frac{\partial u}{\partial x_s} > 0. \tag{4.10}$$

To solve (4.5), Dacorogna and Moser transformed it into the functional equation

$$G(x_s, u(x_s)) = F(x_s) \tag{4.11}$$

where

$$\begin{aligned} G(a,b) &= \int_{R^s_{ab}} g_s(x^s)\,dx^s \\ F(a) &= \int_{Q^s_a} f(x^s)\,dx^s \end{aligned}$$

with

$$\begin{aligned} Q^s_a &= \{x^s \in Q^s \mid 0 < x_s < a\} \\ R^s_{ab} &= \{x^s \in Q^s \mid 0 < x_s < a + \zeta(x^{s-1})b\} \end{aligned} \tag{4.12}$$

where $b \in [-\eta, \eta]$: They obtained this equation by first setting $u(0, x') = 0$ and then integrating the equation (4.5) for $(s-1)$ in place of s, i.e.,

$$\int_{Q^{s-1}} (g_{s-1}(x^{s-1}, x_s, s') - f(x^{s-1}, x_s, x'))\,dx^{s-1} = 0 \tag{4.13}$$

over $0 < x_s < a$, which gives rise to

$$\int_{Q^s_a} (g_{s-1}(x^s, x') - f(x^s, x'))\,dx^s = 0.$$

But this is then equivalent to (4.11).

We note that G (resp. F) is differentiable, if g (resp. f) is continuous. In fact, we have the explicit formulae

$$\frac{\partial G}{\partial b} = \int_{Q^{s-1}} \zeta(x^{s-1}) g_s(x^{s-1}, a + \zeta(x^{s-1})b)\,dx^{s-1} \tag{4.14}$$

$$\frac{\partial G}{\partial a} = \int_{Q^{s-1}} g_s(x^{s-1}, a + \zeta(x^{s-1})b)\,dx^{s-1} \tag{4.15}$$

$$\frac{\partial F}{\partial a} = \int_{Q^{s-1}} f(x^{s-1}, a)\,dx^{s-1}. \tag{4.16}$$

Note that $u(0) = 0$ is the unique solution of (4.11) at $x_s = 0$. At this point, they derived existence and uniqueness of the solution to (4.11) by the intermediate value theorem. We denote by $v = v_{DM}$ and $u = u_{DM}$ for this unique solution and call them Darcorogna-Moser's solution, or simply as DM-solutions.

Remark 4.1. To obtain the C^0 convergence statement (3.12) in Theorem 3.2, we need to control the C^0 distance $\bar{d}(\psi_2, id)$ in the above existence proof of ψ_2. This C^0 estimate is precisely the one left untreated by Dacorogna and Moser in [DM]. However, following Moser's deformation method [Mo] and the use of elliptic second order partial differential equation, they proved an existence of a diffeomorphism ψ_2 satisfying an a priori $C^{k+1,\alpha}$ estimate when $f, g \in C^{k,\alpha}$ *when* $k \geq 1$ *and* $\alpha > 0$ [DM]*. This elliptic approach using the deformation method does not produce the* C^0 *convergence required in (3.12).*

In fact by differentiating (4.11), one obtains

$$\frac{\partial G}{\partial a} + \frac{\partial G}{\partial b}\frac{\partial u}{\partial x_s} = \frac{\partial F}{\partial a}. \tag{4.17}$$

From this, Dacorogna-Moser [DM] derives that the solution u is differentiable. In fact, the standard boot-strap argument, using (4.17) and the fact that the function $\frac{\partial G}{\partial b}$ is positive from (4.14) proves the following a priori $C^{k,\alpha}$ estimate for $k \geq 0$ and

$0 < \alpha < 1$ for the DM-solution itself. This demonstrates that the DM-solution is as good as the one obtained by the deformation approach used in [Mo], [DM] even for the higher regularity.

One main theorem we prove in the current paper is that DM-solutions will also satisfy the additional C^0-coerciveness property under the distance $d_{\mathcal{M}}(m_g, m_f)$ of the weak topology of measures.

For the purpose of our later study of the parametric version of Theorem 3.2, we summarize the above discussion on the higher regularity into the following proposition

Proposition 4.1. *Let g be a given positive $C^{k,\alpha}$ function. Suppose that the functions f is also $C^{k,\alpha}$ and denote by $|\cdots|_{k,\alpha}$ the $C^{k,\alpha}$ norm of functions. Let u be a DM-solution. Then we have*

$$|u|_{k+1,\alpha} \leq C_{(k;g)}|f - g|_{k,\alpha} \tag{4.18}$$

for all $k \geq 0$, where $C_{(k;g)}$ is a constant depending only on k and C^k norm of g.

Remark 4.2. We would like to emphasize that we cannot expect that the derivative of the solution u converges to 0 as $d_{\mathcal{M}}(m_f, m_g) \to 0$. In fact in the above proof, we do not have any control of $|\nabla u|$ in terms of $d_{\mathcal{M}}(m_f, m_g)$.

4.2. **Coercive reformulation.** At the end of the day, one can write Dacorogna-Moser's solution in the form $\psi = \varphi_n \circ \varphi_{n-1} \circ \cdots \circ \varphi_2 \circ \varphi_1$. In coordinate expression $\psi = (v_1, v_2, \cdots, v_n) := v$, v_j has the following form :

$$\begin{aligned}
v_1(x) &= x_1 + u_1(x_1, \widetilde{x}_1) \\
v_2(x) &= x_2 + \zeta_2(v^1(x))u_2(x_2, \widetilde{x}_2) \\
&\vdots \\
v_{n-1}(x) &= x_{n-1} + \zeta_{n-1}(v^{n-2}(x))u_{n-1}(x_{n-1}, x_n) \\
v_n(x) &= x_n + \zeta_n(v^{n-1}(x))u_n(x_n).
\end{aligned} \tag{4.19}$$

Here we denote $\widetilde{x}_i = (x_{i+1}, \cdots, x_n)$ and $v^j = (v_1, \cdots, v_j)$ for $j = 1, \cdots, n$. We would like to *emphasize* that the argument inside ζ_j is $v^{j-1}(x)$, not x^{j-1}.

We will now examine the C^0-behavior of DM-solutions ψ above in terms of the weak topology of measures.

We recall that $\mathrm{supp}(f - g) \subset \mathrm{Int}\, Q$ and so we can choose $\eta > 0$ so that

$$\mathrm{supp}(f - g) \subset \{x \in Q \mid d(x, \partial Q) \geq \eta\} = Q^n(1 - \eta). \tag{4.20}$$

This choice of η depends only on $\mathrm{supp}(f - g)$, independent of individual f or g. The choice of η will be fixed for the rest of the paper. Without loss of generality, we also assume that f, g are indeed defined on the bigger cube $Q^n(1 + \eta)$ where

$$Q^n(1 + \eta) = \{x \in \mathbb{R}^n \mid -\eta \leq x_j \leq 1 + \eta,\, j = 1, \cdots, n\}.$$

We now fix a family of cut-off functions $\zeta = \{\zeta_s\}_{s=2}^n$ with

$$\zeta_s : Q^{s-1} \to \mathbb{R} \quad \text{with } \mathrm{supp}\, \zeta_s \subset Q^{s-1}(1 - \tfrac{\eta}{2}) \tag{4.21}$$

for $s = 2, \cdots, n$ such that

$$\begin{cases} 0 \leq \zeta_s \leq 1 + \varepsilon_0 \quad \text{in } Q^n \\ \int_{Q^{s-1}} \zeta_s(x^{s-1})\, dx^{s-1} = 1 \\ \int_{Q^{s-1}} |\zeta_s(x^{s-1}) - 1|\, dx^{s-1} < \varepsilon_0 \end{cases} \tag{4.22}$$

as in (4.8) where $\varepsilon_0 = \varepsilon_0(\zeta) > 0$ is a constant, which satisfies

$$\varepsilon_0(\zeta) < \min\left\{ \frac{\min\{\min f, \min g\}}{\max g}, \frac{\min f}{2 \max g} \right\}. \tag{4.23}$$

This constant $\varepsilon_0(\zeta)$ can be made as small as we want independently of the given g, f. (See Remark 3.1 (2) and (3).) For example, we can always choose

$$\varepsilon_0(\zeta) < d_{\mathcal{M}}(m_g, m_f). \tag{4.24}$$

Motivated by the expression given in (4.19), we introduce the following definition which will be essential for our discussion following afterwards.

Definition 4.3. We call a map $u : Q^n \to \mathbb{R}^n$ *triangular* if its components u_j have the following triangular form :

$$\begin{aligned} u_1 &= u_1(x_1, \cdots, x_n) \\ u_2 &= u_2(x_2, \cdots, x_n) \\ &\vdots \\ u_{n-1} &= u_{n-1}(x_{n-1}, x_n) \\ u_n &= u_n(x_n). \end{aligned}$$

We denote by $C^0_{tri}(Q^n, \mathbb{R}^n)$ the set of triangular maps. We define

$$B \subset C^0_{tri}(Q^n, \mathbb{R}^n)$$

the set of triangular maps satisfying $u(1, \cdots, 1) = 0$.

Obviously B is a closed subspace of the Banach space $C^0(Q^n, \mathbb{R}^n)$ and hence itself a Banach space with the C^0-norm

$$|u| = |u|_{C^0} = \max_{1 \leq j \leq n} |u_j|$$

for the vector map $u = (u_1, \cdots, u_n)$. Furthermore it follows from this triangularity of u that the Jacobian ∇u of u forms an upper triangular matrix.

Now the DM-solutions $v = v_{DM} : Q^n \to Q^n$ have the following form

$$v(x_1, \cdots, x_n) = (v_1, \cdots, v_j, \cdots, v_n) \tag{4.25}$$

where $v_j : Q \to [0, 1]$ is a function of the type

$$v_j(x) = v_j(x_j, \widetilde{x}_j) = x_j + \zeta_j(v^{j-1}(x)) u_j(x_j, x'), \quad j = 2, \cdots, n \tag{4.26}$$

$$v_1(x) = v_1(x_1, \cdots x_n) = x_1 + u_1(x_1, \cdots, x_n) \tag{4.27}$$

with $\widetilde{x}_j = (x_{j+1}, \cdots, x_n)$. In other words, we can factorize ψ_2 into

$$\psi_2 = \varphi_n \circ \varphi_{n-1} \circ \cdots \circ \varphi_1$$

where each φ_j is a smooth map of the form given in (4.6) depending on u.

Then the diffeomorphism v satisfies $g(v(x)) \det \nabla v(x) = f(x)$ and its weak form

$$\int_{v(E)} g(y)\, dy = \int_E f(x)\, dx \tag{4.28}$$

for any measurable subset E. We define

$$\begin{aligned} Q^n_{a;j} &= \{x \in Q^n \mid 0 \leq x_i \leq 1 \text{ for } 1 \leq i \leq j-1, \\ &\qquad 0 \leq x_i \leq a_i,\ j \leq i \leq n\} & (4.29) \\ R^n_{au;j} &= v(Q^n_{a;j}) & (4.30) \end{aligned}$$

for $j = 1, \cdots, n$.

Knowing that the DM-solution $v = \psi_2$ is a homeomorphism (in fact a smooth diffeomorphism when g, f are smooth), $R^n_{au;j}$ is a closed measurable subset and so we can define the integrals

$$\begin{aligned} G_j(a;u) &= \int_{R^n_{au;j}} g(y)\, dy & (4.31) \\ F_j(a) &= \int_{Q^n_{a;j}} f(x)\, dx & (4.32) \end{aligned}$$

and consider the vector functions

$$G = (G_1, \cdots, G_n), \qquad F = (F_1, \cdots, F_n)$$

where we denote $G := G(\cdot; u)$. Then the weak form (4.11) of the equation

$$g(v(x)) \det \nabla v(x) = f(x)$$

can be reduced to (4.33)

$$G(a;u) = F(a), \quad a \in Q^n. \tag{4.33}$$

In particular, DM-solution satisfies (4.33).

The converse also holds for differentiable maps.

Lemma 4.2. *If u is a solution of (4.33) that is differentiable, then it satisfies*

$$g(v(x)) \det \nabla v(x) = f(x). \tag{4.34}$$

Proof. Since u is differentiable, we can apply the change of variables and rewrite (4.33) as

$$\int_{Q^n_{a;j}} g(\psi_2(x)) \nabla \psi_2(x)\, dx = \int_{Q^n_{a;j}} f(x)\, dx$$

for all $j = 1, \cdots, n$. The lemma then follows by taking the partial derivatives of these equations with respect to a_i for each $i = 1, \cdots, n$. $\square$

Now we consider the subset $B_{homeo} \subset B$ defined by

$$B_{homeo} = \{u \in B \mid \text{the associated map } v \text{ in (4.26) and (4.27) is a homeomorphism}\}.$$

Then for each element $u \in B_{homeo}$, the functions G_j are defined and so we can define a map

$$\Psi : B_{homeo} \to C^0(Q^n, \mathbb{R}^n) \tag{4.35}$$

by $\Psi = (\Psi_1, \cdots, \Psi_n)$ whose components are given by

$$\Psi_j(u) = G_j(\cdot; u) - F_j(\cdot).$$

We remark that the equation (4.33) is equivalent to $\Psi(u) = 0$.

The following proposition is the reason why we introduce the notion of triangular maps and the space B.

Proposition 4.3. *The map $a \mapsto F(a)$ is triangular, and so is $a \mapsto G(a; u)$ whenever $u \in B_{homeo}$. In particular, the map Ψ maps B_{homeo} to B.*

Proof. Recall the definitions of G and F in (4.31) and (4.32) respectively. By the definition (4.29) of $Q^n_{a;s}$, it does not depend on $a_1, \cdots, a_{s-1}$ and hence neither does $R^n_{au;s} = v(Q^n_{a;s})$. This immediately implies that both F and $G(\cdot; u)$ are triangular. This finishes the proof of triangularity of Ψ.

We next check $\Psi(u)(1, \cdots, 1) = 0$. Since $u(1, \cdots, 1) = 0$ we have

$$R^n_{(\vec{1}u;n)} = R^n_{(\vec{1}\vec{0};n)} = Q^n.$$

where $\vec{1} = (1, \cdots, 1)$ and $\vec{0} = (0, \cdots, 0)$. Therefore we have

$$\Psi(u)(1, \cdots, 1) = \int_{Q^n} g\, dy - \int_{Q^n} f\, dx$$

which is assumed to be zero in (3.10). This finishes the proof. $\square$

5. Linearization

Now we introduce the subset $B_{diff} \subset B_{homeo}$ consisting of smooth maps u whose associated map v is a diffeomorphism. Then the restriction of Ψ to B_{diff} is continuously differentiable map to $C^\infty(Q^n, \mathbb{R}^n)$ in the Frechet sense : Since diffeomorphism property of a map defined on compact sets is an open property, once we know that B_{diff} is non-empty, it is an open subset of $C^\infty(Q^n, \mathbb{R}^n)$ and hence we can define the Frechet derivative of Ψ on B_{diff}.

Denote by $\overline{0}$ the zero function. We now compute the Frechet derivative of $\Psi_{B_{diff}}$ at $u = \overline{0} \in B_{diff}$ which corresponds to $v = id$.

Applying the Taylor expansion to Ψ at $u = \overline{0}$, (4.11) can be rewritten as

$$-\, d\Psi(\overline{0}) \cdot u = \Psi(\overline{0}) + N(u) \tag{5.1}$$

where $d\Psi$ is the Frechet derivative of

$$\Psi : B_{diff} \to C^\infty(Q^n, \mathbb{R}^n)$$

and

$$N(u) = \Psi(u) - \Psi(\overline{0}) - d\Psi(\overline{0}) \cdot u$$

is the 'higher order term'. It follows from the definitions of Ψ_j that we have

$$\begin{aligned} \Psi_j(\overline{0})(a) &= G_j(a; \overline{0}) - F_j(a) \\ &= \int_{Q^n_{a;j}} g(y) dy - \int_{Q^n_{a;j}} f(x) dx. \end{aligned} \tag{5.2}$$

Now the following provides an explicit formula for the Frechet derivative of the map

$$d\Psi(\overline{0}) : C^\infty_{tri}(Q^n, \mathbb{R}^n) \to C^\infty_{tri}(Q^n, \mathbb{R}^n)$$

at $u = \overline{0}$.

Proposition 5.1. *Let* $X = (X_1, \cdots, X_n) \in C^\infty_{tri}(Q^n, \mathbb{R}^n)$. *Then*

$$(d\Psi(\overline{0}) \cdot X)_j(a) = \sum_{k=j}^n \int_{Q^{n-k}_{a;j}} X_k(a_k, \widetilde{x}_k)\Big(\int_{Q^{k-1}_{a;j,k}} \zeta_j(x^{k-1}) g(x^{n-1}_{a;k}) dx^{k-1}\Big) d\widetilde{x}_k. \quad (5.3)$$

In particular, the matrix elements

$$(d\Psi(\overline{0}))_{jk} : C^\infty(Q^{n-k}, \mathbb{R}) \to C^\infty(Q^{n-j}, \mathbb{R})$$

of the matrix operator

$$d\Psi(\overline{0}) : C^\infty_{tri}(Q^n, \mathbb{R}^n) \to C^\infty_{tri}(Q^n, \mathbb{R}^n)$$

are given by

$$\big((d\Psi(\overline{0}))_{jk}(h)\big)(a) = \int_{Q^{n-k}_{a;j}} C_{jk}(a_j, \cdots, a_k, \widetilde{x}_k) h(\widetilde{x}_k)\, d\widetilde{x}_k \quad (5.4)$$

where $C_{jk}(a_j, \cdots, a_k, \widetilde{x}_k)$ *are smooth functions of* $(a_j, \cdots, a_k, \widetilde{x}_k)$ *defined by*

$$C_{jk}(a_j, \cdots, a_k, , \widetilde{x}_k) = \begin{cases} \int_{Q^{n-1}_{a;j}} \zeta_n(x^{n-1}) g(x^{n-1}, a_n) dx^{n-1} & \text{for } k = n \\ \int_{Q^{k-1}_{a;j}} \zeta_k(x^{k-1}) g(x^{k-1}, a_k, \widetilde{x}_k) dx^{k-1} & \text{for } j \leq k \leq n-1 \\ 0 & \text{for } k < j \end{cases} \quad (5.5)$$

Proof. Recall $v = \varphi_n \circ \cdots \circ \varphi_1$ and

$$\varphi_j(x_1, \cdots, x_n) = (x_1, \cdots, x_j + \zeta_j(x^{j-1}) u(x_j, \widetilde{x}_j), \cdots, x_n). \quad (5.6)$$

We also note that we can write

$$\int_{v(Q^n_{a;j})} g dy = \int_{Q^n_{a;j}} v^*(g\, dy)$$

where

$$v^*(g\, dy) = \varphi_1^* \circ \cdots \circ \varphi_n^*(g\, dy).$$

Therefore to compute $d\Psi(\overline{0}) \cdot X$, we need to first compute the variation $\delta\varphi_j(X)$. But it is easy to see from definition (5.6) of φ_j

$$\delta\varphi_j(X) = (\zeta_j X_j) \frac{\partial}{\partial x_j} \quad (5.7)$$

and so

$$(d\Psi(\overline{0}) \cdot X)_j(a) = \sum_{k=1}^n \int_{Q^n_{a;j}} \mathcal{L}_{\delta\varphi_k(X)}(g\, dx) = \sum_{k=1}^n \int_{Q^n_{a;j}} d(\delta\varphi_k(X) \rfloor (g\, dx)).$$

On the other hand from the definition of $Q^n_{a;j}$, the triangularity of X and (5.7), the latter identity becomes

$$\begin{aligned}
(d\Psi(\overline{0})\cdot X)_j(a) &= \sum_{k=j}^n \int_{Q^n_{a;j}} d(\delta\varphi_k(X)\rfloor(g\,dx)) \\
&= \sum_{k=j}^n \int_{Q^n_{a;j}} \frac{\partial}{\partial x_k}(g\zeta_k X_k)dx_k dx^{n-1}_k \\
&= \sum_{k=j}^n \int_{Q^{n-1}_{a;j,k}} g(x^{n-1}_{a;k})\zeta_k(x^{k-1})X_k(a_k,\widetilde{x}_k)dx^{n-1}_k \\
&= \sum_{k=j}^n \int_{Q^{n-k}_{a;j}} X_k(a_k,\widetilde{x}_k)\Big(\int_{Q^{k-1}_{a;j}} \zeta_k(x^{k-1})g(x^{n-1}_{a;k})dx^{k-1}\Big)d\widetilde{x}_k.
\end{aligned}$$

Here we define the $(n-1)$-vectors

$$x^{n-1}_{a;k} = (x^{k-1}, a_k, \widetilde{x}_k), \quad \text{for } k \geq j \tag{5.8}$$

and denote the volume element of any of $x^{n-1}_{a;k}$ by dx^{n-1}_k. Then the third equality above follows by integration by parts over x_k. This finishes the proof. □

Next we introduce the following function space which will be essential for the later discussions :

Definition 5.1. We define

$$C^{\vec{1}}_{tri}(Q^n,\mathbb{R}^n)$$

to be the set of continuous triangular maps $f \in C^0_{tri}(Q^n,\mathbb{R}^n)$ whose components are given by the functions $f_j : Q^{n-j} \to \mathbb{R}$ such that

$$D^\alpha f : Q^{n-j} \to \mathbb{R}, \quad Q^{n-j} = \{(x_{j+1},\cdots,x_n) \mid 0 \leq x_l \leq 1,\, l = j+1,\cdots,n\}$$

are continuous for any subset $\alpha \subset \{j+1,\cdots,n\}$. Here $\vec{1}$ stands for $\vec{1} = (1,\cdots,1)$ and $D^\alpha f$ for the partial derivative with respect to the multi-index α.

It is easy to check that $C^{\vec{1}}_{tri}(Q^n,\mathbb{R}^n)$ becomes a Banach space if we equip it with a norm given by

$$\|f\| = \max_{j=1,\cdots,n}\{\|f_j\|_{C^{\vec{1}}}\}$$

where $\|f_j\|_{C^{\vec{1}}}$ is given by

$$\|f_j\|_{C^{\vec{1}}} = \max_{\alpha\subset\{j+1,\cdots,n\}} |D^\alpha f_j|_{C^0}. \tag{5.9}$$

We recall that for any function $f \in C^{\vec{1}}_{tri}(Q^n,\mathbb{R}^n)$ the partial derivatives $D^\alpha f$ does not depend on the ordering of indices contained in the subset $\alpha \subset \{j+1,\cdots,n\}$ (See Theorem 7.3 [La].)

With this preparation, we now prove

Proposition 5.2. *$d\Psi(\overline{0})$ continuously extends to a bounded linear operator from $C^0_{tri}(Q^n,\mathbb{R}^n)$ to $C^{\vec{1}}_{tri}(Q^n,\mathbb{R}^n)$ which is bijective. Denote the extension again by*

$$d\Psi(\overline{0}) : C^0_{tri}(Q^n,\mathbb{R}^n) \to C^{\vec{1}}_{tri}(Q^n,\mathbb{R}^n).$$

In particular, it is invertible. We denote its inverse by

$$(d\Psi(\overline{0}))^{-1} : C^{\vec{1}}_{tri}(Q^n,\mathbb{R}^n) \to C^0_{tri}(Q^n,\mathbb{R}^n). \tag{5.10}$$

Proof. From the matrix expression (5.3) of $d\Psi(\overline{0}) = \big(d\Psi(\overline{0})_{jk}\big)$, we see that it becomes a triangular matrix and is represented by the integral pairing with the functions C_{jk} and manifestly extends to an operator from $C^0(Q^{n-k},\mathbb{R})$ to $C^{\vec{1}}(Q^{n-j},\mathbb{R})$.

And once we have proved the bijectivity of the bounded linear operator $d\Psi(\overline{0})$, the open mapping theorem will imply that the operator is invertible. Therefore it remains to prove bijectivity.

We start with the proof of injectivity. Suppose that $d\Psi(\overline{0})(X) = 0$ for $X = (X_1,\cdots,X_n) \in C^0_{tri}(Q^n,\mathbb{R}^n)$. By (5.3) and (5.5), X satisfies

$$\sum_{k=j}^n \int_{Q^{n-k}_{a;j}} X_k(a_k,\widetilde{x}_k) C_{jk}(a_j,\cdots,a_k,\widetilde{x}_k)\, d\widetilde{x}_k = 0 \tag{5.11}$$

for all $j = 1,\cdots,n$. We will prove $X = 0$ by a downward induction over j. First consider the term for $j = n$. In this case, this reduces to

$$0 = X_n(a_n)C_{nn}(a_n).$$

Since $C_{nn}(a_n) = \int_{Q^{n-1}_{a;n,n}} \zeta_n(x^{n-1})g(x^{n-1},a_n)dx^{n-1} > 0$, we derive $X_n \equiv 0$.

Now suppose we have shown

$$X_n = X_{n-1} = \cdots = X_{l+1} = 0$$

and consider the equation $(d\Psi(\overline{0})(X))_l = 0$. Under this assumption, (5.11) for $j = l$ reduces to

$$\int_{Q^{n-l}_{a;l}} X_l(a_l,\widetilde{x}_l)C_{ll}(a_l,\widetilde{x}_l)d\widetilde{x}_l = 0$$

for all $a \in Q^n$. Differentiating this identity with respect to a_j successively for $j = l+1,\cdots,n$ at the vector $a = (a_1,\cdots,a_n)$, we obtain

$$0 = X_l(a_l,\widetilde{a}_l)C_{ll}(a_l,\widetilde{a}_l) = X_l(a)C_{ll}(a_l,\widetilde{a}_l).$$

Since $C_{ll}(a_l,\widetilde{a}_l) > 0$, we obtain $X_l \equiv 0$ as before. This proves injectivity of $d\Psi(\overline{0})$.

Now we turn to surjectivity thereof. Let $Y = (Y_1,\cdots,Y_n) \in C^{\vec{1}}_{tri}(Q^n,\mathbb{R}^n)$ and consider the equation

$$d\Psi(\overline{0})(X) = Y \quad \text{or equivalently } \ (d\Psi(\overline{0})(X))_j = Y_j,\ j = 1,\cdots,n$$

for $X \in C^0_{tri}(Q^n,\mathbb{R}^n)$. Again we solve this by downward induction starting from $j = n$. For $j = n$, this reduces to

$$X_n(a_n)C_{nn}(a_n) = Y_n(a_n)$$

and so obtain $X_n(a_n) = Y_n(a_n)/C_{nn}(a_n)$. Now suppose that we have solved for $j = n,\cdots,l+1$, and consider the equation $(d\Psi(\overline{0})(X))_l = Y_l$. This equation becomes

$$\sum_{k=l}^n \int_{Q^{n-k}_{a;j}} X_k(a_k,\widetilde{x}_k)C_{lk}(a_l,\cdots,a_k,\widetilde{x}_k)d\widetilde{x}_k = Y_l(a_l,\cdots,a_n). \tag{5.12}$$

Since $Y_l \in C^{\vec{1}}(Q^{n-l},\mathbb{R})$, we can differentiate this equation with respect to a_j successively over $j = l+1,\cdots,n$ and obtain

$$X_l(a_l,\widetilde{a}_l)C_{ll}(a_l,\widetilde{a}_l) + \sum_{k=l+1}^n X_k(a_k,\widetilde{a}_k)\frac{\partial C_{lk}}{\partial a_{l+1}\partial a_{l+2}\cdots\partial a_k} = \frac{\partial Y_l}{\partial a_{l+1}\cdots\partial a_n}$$

by the triangularity of X and Y. Since $C_{ll}(a_l, \widetilde{a}_l) > 0$, we obtain

$$\begin{aligned} X_l(a) &= X_l(a_l, \widetilde{a}_l) \\ &= \frac{1}{C_{ll}(a_l, \widetilde{a}_l)} \left(\frac{\partial Y_l}{\partial a_{l+1} \cdots \partial a_n}(a) - \sum_{k=l+1}^{n} X_k(a) \frac{\partial C_{lk}}{\partial a_{l+1} \partial a_{l+2} \cdots \partial a_k}(a) \right). \end{aligned}$$

We note that by the induction hypothesis, the right hand side is already determined. Since $Y_l \in C^{\vec{1}}(Q^{n-l}, \mathbb{R})$ and C_{lk} are smooth, the right hand side is continuous and hence lies in $C^0_{tri}(Q^n, \mathbb{R}^n)$. This finishes the induction step and so solves the equation $d\Psi(\overline{0})(X) = Y$ for any $Y \in C^{\vec{1}}_{tri}(Q^n, \mathbb{R}^n)$ and so finishes the proof of surjectivity. Hence the proof. □

The following proposition is a crucial ingredient which saves us from doing derivative estimates for the nonlinear terms in section 9.

Proposition 5.3. *The operator $(d\Psi(\overline{0}))^{-1}$ given in (5.10) continuously extends to a bounded linear operator*

$$K_g : C^0_{tri}(Q^n, \mathbb{R}^n) \to C^0_{tri}(Q^n, \mathbb{R}^n).$$

Proof. We go back to the surjectivity proof of Proposition 5.2. It will be enough to prove that there exists a constant $M_g > 0$ such that the unique solution X for $d\Psi(\overline{0})(X) = Y$ satisfies

$$|X|_{C^0} \leq M_g |Y|_{C^0} \tag{5.13}$$

for any given $Y \in C^{\vec{1}}_{tri}(Q^n, \mathbb{R}^n)$. We prove this again by the downward induction.

For $j = n$, we have $X_n(a_n) C_{nn}(a_n) = Y_n(a_n)$ and hence

$$|X_n(a_n)| \leq \frac{|Y_n(a_n)|}{\min_{a_n} C_{nn}(a_n)}$$

recalling $C_{nn}(a_n) \neq 0$ from (5.5). But we have

$$C_{nn}(a_n) = \int_{Q^{n-1}_{a;n}} \zeta_n(x^{n-1}) g(x^{n-1}, a_n) dx^{n-1}$$

from (5.5). In particular, we have

$$\min_{a_n} C_{nn}(a_n) \geq \min g \int_{Q^{n-1}_{a;n}} \zeta_n(x^{n-1}) dx^{n-1} = \min g$$

where we use (4.22) for the equality. Therefore we have proved

$$|X_n|_{C^0} \leq \frac{1}{\min g} |Y_n|_{C^0} \tag{5.14}$$

for all $l+1 \leq j \leq n$. Now as the induction hypothesis, suppose that there exists a constant $M_{\ell+1}$ such that

$$|X_j|_{C^0} \leq M_{\ell+1} \max_{\ell+1 \leq j \leq n} |Y_j|_{C^0} \tag{5.15}$$

for all $l+1 \leq j \leq n$. We rewrite (5.12) into

$$\begin{aligned} \int_{Q^{n-l}_{a;l}} X_l(a_l, \widetilde{x}_l) C_{ll}(a_l, \widetilde{x}_l) d\widetilde{x}_l &= Y_l(a_l, \cdots, a_n) \\ - \sum_{k=l+1}^{n} \int_{Q^{n-k}_{a;j}} & X_k(a_k, \widetilde{x}_k) C_{lk}(a_l, \cdots, a_k, \widetilde{x}_k) d\widetilde{x}_k. \end{aligned} \tag{5.16}$$

By the induction hypothesis the sum in the right hand side can be estimated as

$$\begin{aligned}
&\left|\sum_{k=l+1}^{n}\int_{Q^{n-k}_{a;j}} X_k(a_k,\widetilde{x}_k)C_{lk}(a_l,\cdots,a_k,\widetilde{x}_k)d\widetilde{x}_k\right| \\
\leq\ & \int_{Q^{n-k}_{a;j}} M_{\ell+1}\max_{\ell+1\leq j\leq n}\{|Y_j|_{C^0}\}C_{lk}(a_l,\cdots,a_k,\widetilde{x}_k)d\widetilde{x}_k \\
\leq\ & M_{\ell+1}\max_{\ell+1\leq j\leq n}\{|Y_j|_{C^0}\}\int_{Q^{n-k}_{a;j}} C_{lk}(a_l,\cdots,a_k,\widetilde{x}_k)d\widetilde{x}_k.
\end{aligned}$$

But we derive

$$\begin{aligned}
\int_{Q^{n-k}_{a;j}} C_{lk}(a_l,\cdots,a_k,\widetilde{x}_k)d\widetilde{x}_k &= \int_{Q^{n-k}_{a;\ell}}\left(\int_{Q^{k-1}_{a;\ell}}\zeta_k(x^{k-1})g(x^{k-1},a_k,\widetilde{x}_k)dx^{k-1}\right)d\widetilde{x}_k \\
&\leq \max g\left(\int_{Q^{k-1}_{a;\ell}}\zeta_k(x^{k-1})dx^{k-1}\right)\leq \max g
\end{aligned}$$

again using (4.22). Hence we have obtained

$$\begin{aligned}
&\left|\sum_{k=l+1}^{n}\int_{Q^{n-k}_{a;j}} X_k(a_k,\widetilde{x}_k)C_{lk}(a_l,\cdots,a_k,\widetilde{x}_k)d\widetilde{x}_k\right| \\
\leq\ & (n-\ell-1)M_{\ell+1}\max g\max_{\ell+1\leq j\leq n}\{|Y_j|_{C^0}\}.
\end{aligned}$$

Substituting this into (5.16), we obtain

$$\begin{aligned}
&\left|\int_{Q^{n-l}_{a;l}} X_l(a_l,\widetilde{x}_l)C_{ll}(a_l,\widetilde{x}_l)d\widetilde{x}_l\right| \\
\leq\ & |Y_l|_{C^0}+(n-\ell-1)M_{\ell+1}\max g\max_{\ell+1\leq j\leq n}\{|Y_j|_{C^0}\} \\
\leq\ & (n-\ell)\max\{M_\ell,1\}\max g\max_{\ell\leq j\leq n}\{|Y_j|_{C^0}\}\}.
\end{aligned}$$

On the other hand, the left hand side can be estimated from below

$$\begin{aligned}
\left|\int_{Q^{n-l}_{a;l}} X_l(a_l,\widetilde{x}_l)C_{lk}(a_l,\widetilde{x}_l)d\widetilde{x}_l\right| &\geq |X_l(a_l,\widetilde{x}_l)|\min C_{ll} \\
&\geq |X_l(a_l,\widetilde{x}_l)|\min g.
\end{aligned}$$

Combining the last two inequalities, we have obtained

$$|X_l(a_l,\widetilde{x}_l)|\leq (n-\ell)\frac{\max\{M_{\ell+1},1\}\max g\max_{\ell\leq j\leq n}\{|Y_j|_{C^0}\}}{\min g}.$$

By defining

$$M_\ell=(n-\ell)\frac{\max\{M_{\ell+1},1\}\max g}{\min g}$$

we have finished the induction step and hence the proof of (5.13).

In fact the above proof shows that M_g can be chosen to be

$$M_g=n!\max\left\{\frac{1}{\min g}\left(\frac{\max g}{\min g}\right)^{n-1},1\right\} \tag{5.17}$$

and hence we have $\|K_g\|\leq M_g$. This finishes the proof. □

We recall that the constant M_g does not increase under the reduction process to a smaller cubes by the reasons mentioned in Remark 3.1.

6. C^0-coerciveness of Darcorogna-Moser's solutions

We denote the operator norm of the bounded linear operator K_g given in Proposition 5.3 by $\|K_g\|$ which has the bound

$$\|K_g\| \leq M_g \tag{6.1}$$

where M_g is the constant given in (5.17). From the explicit formula of M_g, it follows that M_g depends only on g and is continuous on g in C^0-topology.

In this section, all the norms $|\cdot|$ below will denote the C^0-norms.

We write (5.1) in the following form

$$u = \Xi(u) \tag{6.2}$$

where Ξ is the map from $B \to B$ defined by

$$\Xi(u) = -(d\Psi(\overline{0}))^{-1}(\Psi(\overline{0}) + N(u)). \tag{6.3}$$

Here we would like to note from (5.2) that $\Psi(\overline{0})$ lies in $C^{\vec{1}}(Q^n, \mathbb{R}^n)$. On the other hand, we can rewrite $N(u)$

$$\begin{aligned} N_j(u) &= \Psi_j(u) - \Psi_j(\overline{0}) - (d\Psi(\overline{0}) \cdot u)_j \\ &= \int_{R^n_{au;j}} g(y)dy - \int_{Q^n_{a;j}} g(y)dy \\ &\quad - \sum_{k=j}^{n} \int_{Q^{n-k}_{a;j}} u_k(a_k, \widetilde{x}_k)\Big(\int_{Q^{k-1}_{a;j,k}} \zeta_j(x^{k-1}) g(x^{n-1}_{a;k}) dx^{k-1}\Big) d\widetilde{x}_k \end{aligned} \tag{6.4}$$

From this, it follows that $N(u)$ also lies in $C^{\vec{1}}_{tri}(Q^n, \mathbb{R}^n)$ if u is smooth as for $u = u_{DM}$. Therefore $\Psi(\overline{0}) + N(u)$ lies in the domain of $(d\Psi(\overline{0}))^{-1}$ and hence the expression (6.3) is well-defined for $u = u_{DM}$.

We derive from (6.2), (6.3)

$$|u| \leq |(d\Psi(\overline{0}))^{-1}(\Psi(\overline{0}) + N(u))| \leq M_g(|\Psi(\overline{0})| + |N(u)|). \tag{6.5}$$

We now estimate $|\Psi(\overline{0})|$ and $|N(u)|$ separately.

We start with the following

Proposition 6.1. *We have*

$$|\Psi(\overline{0})| \leq d_{\mathcal{M}}(m_f, m_g) \tag{6.6}$$

where $m_f := \mu_{(fdx)}$ is the measure associated to the volume form fdx and similarly for m_g.

Proof. From (5.2), we have for the j-th component of the vector $\Psi(\overline{0})(a)$

$$\Psi(\overline{0})_j(a) = G_j(a, 0) - F_j(a) = \int_{Q^n_{a;j}} g(y)dy - \int_{Q^n_{a;j}} f(x)dx. \tag{6.7}$$

On the other hand $R_{a;j} = Q_{a;j}$ for $u = \overline{0}$. Therefore from the definition (2.1) of the metric $d_{\mathcal{M}} = \mathrm{Lid}_1$, we have derived the upper-bound for the 'zero-order term'

$$|\Psi(\overline{0})| \leq d_{\mathcal{M}}(m_f, m_g) \tag{6.8}$$

where we use the fact that the integral (6.7) corresponds to

$$\int \chi_{Q^n_{a;j}}\, dm_g - \int \chi_{Q^n_{a;j}}\, dm_f$$

which is obtained by taking the characteristic function $\chi_{Q^n_{a;j}}$ of $Q^n_{a;j}$ as the test function in (2.1) for $b = 1$. □

Proposition 6.1 is a place where *the weak topology of measures enters in our proof of C^0-coerciveness* (1.2) in Theorem II. The other such places appearing later will be similar to this one.

Next we do estimates of $N(u)$. For this purpose, we introduce the constant

$$\varepsilon_1(\zeta) = \max_{1\leq k\leq n} \left(\int_{Q^{k-1}} |1 - \zeta_k(y^{k-1})| dy^{k-1} \right). \tag{6.9}$$

See the end of section 9 for our motivation for considering this constant where it appears in middle of the main technical estimates. We like to emphasize that *this constant can be made as small as we want by approximating ζ L^1-close to the function* 1*, once g, f are given.* In particular, we may assume

$$\varepsilon_1(\zeta) < d_{\mathcal{M}}(m_f, m_g). \tag{6.10}$$

Next using the continuity of g and compactness of Q^n, we have the Lipschiz bound

$$|g(x) - g(y)| \leq L_g \cdot |x - y| \tag{6.11}$$

for a constant $L_g > 0$ depending only on g. In fact, L_g is nothing but the *modulus of continuity* of g.

The following is a key lemma whose proof we postpone until section 9 because the proof is rather long and complicated. The main reason behind the presence of this kind of estimates is that $N(u)$ is the higher order term in the Taylor expansion of Ψ. However, since we need to know the precise form of the inequality with respect to g, we need to carry out rather delicate estimates.

Lemma 6.2. *Define*

$$C_4 = C_4(g) = \max\{8 \max g, 4\}$$

and let $u = u_{DM}$ be a DM-solution. Then we have the inequality

$$|N(u)| \leq C_4 \cdot (|\zeta \cdot u| + \varepsilon_1(\zeta) + L_g|\zeta \cdot u|)\, |u|, \tag{6.12}$$

where we denote $\zeta \cdot u := (u_1, \zeta_2 u_2, \cdots, \zeta_n u_n)$

Combining Proposition (6.8), (6.12), and (6.5), we obtain

$$|u| \leq M_g(d_{\mathcal{M}}(m_f, m_g) + C_4 \cdot (|\zeta \cdot u| + \varepsilon_1(\zeta) + L_g|\zeta \cdot u|)\, |u|). \tag{6.13}$$

We can choose the functions $\zeta = \{\zeta_k\}_{k=2,\cdots,n}$ so that

$$\varepsilon_1(\zeta) < d_{\mathcal{M}}(m_f, m_g)$$

as mentioned in (6.10). Substituting this into and rewriting (6.13), we obtain

$$\Big(1 - C_4 \cdot (|\zeta \cdot u| + d_{\mathcal{M}}(m_f, m_g) + L_g \cdot |\zeta \cdot u|)\Big)|u| \leq M_g \cdot d_{\mathcal{M}}(m_f, m_g). \tag{6.14}$$

At this stage, we recall that a DM-solution has the form

$$v = (v_1, v_2, \cdots, v_n)$$

with $v_j(x) = x_j + \zeta_j(v^{j-1}(x))u_j(x_j, \widetilde{x}_j)$ and maps Q^n into Q^n. In particular, we have

$$|\zeta_j u_j| = \max_{x\in Q^n} |\zeta_j(x)u_j(x)| \leq 2.$$

We would also like to emphasize that the constants $C_4(g)$ and L_g depend only on g but not on f, except in the loose way mentioned in (4.23). Therefore Dacorogna-Moser's construction of solution can be equally carried out for the maps defined on the cube $Q^n(K)$ with any length $0 < K \leq 1$ of its sides with the same constants C_4 and L_g. In that case, all DM-solutions on $Q^n(K)$ will satisfy

$$|\zeta \cdot u| \leq 2K$$

because u maps $Q^n(K)$ to $Q^n(K)$ in that case.

Therefore if we set $K = 2^{-N_0}$ and fix $N_0 = N_0(g) \in \mathbb{N}$ such that

$$C_4 \cdot (1 + L_g) \cdot 2^{-(N_0-1)} < \frac{1}{4} \tag{6.15}$$

and consider a DM-solution on the cube $Q^n(K)$, we will have

$$C_4 \cdot (2^{-(N_0-1)} + d_{\mathcal{M}}(m_f, m_g) + L_g \cdot 2^{-(N_0-1)}) < \frac{1}{2} \tag{6.16}$$

for any f such that

$$C_4 d_{\mathcal{M}}(m_f, m_g) < \frac{1}{4}.$$

We recall from Remark 3.1 that this inequality will be achieved by considering μ_σ, μ_τ with $d_{\mathcal{M}}(\mu_\sigma, \mu_\tau) \to 0$ on the original space X given in Theorem II.

Then (6.14) and (6.16) imply

$$|u| \leq 2M_g \cdot d_{\mathcal{M}}(m_f, m_g).$$

Here we recall from (3.7) that M_g depends only on the originally given function g defined on the unit cube Q^n.

Now by decomposing Q^n into cubes of size 2^{-N_0}, and applying this inequality uniformly over to each of the cubes, we obtain the following proposition. Here N_0 is the integer chosen as in (6.15), which depends only on the originally given function g defined on Q^n.

Proposition 6.3. *Let g be a positive continuous function and denote by $m_g = \mu_{(gdx)}$ the associated measure on Q^n. Consider the Darcorogna-Moser's solution $u = u_{DM}$ corresponding to f satisfying the hypotheses in Theorem 3.2. Then there exists a continuous function $r = r(t; g)$ of t, depending only on g, such that $r \to 0$ as $t \to 0$ for which the following holds :*

$$|u|_{C^0} \leq r(d_{\mathcal{M}}(m_f, m_g); g). \tag{6.17}$$

To wrap-up the proof of statement (1) of Theorem 3.2, we need to estimate $d_{C^0}(\psi_2^{-1}, id)$ for $\psi_2 = v$.

For the estimate of $d_{C^0}(\psi_2^{-1}, id)$, we derive $d_{C^0}(\psi_2^{-1}, id) = d_{C^0}(id, \psi_2)$. For we have

$$d_{C^0}(\psi_2^{-1}, id) = \max_{x\in Q^n} d(\psi_2^{-1}(x), x) = \max_{x\in Q^n} d(\psi_2^{-1}(x), \psi_2(\psi_2^{-1}(x)) \leq d_{C_0}(id, \psi_2)$$

and prove the opposite inequality in the same way. Therefore we obtain

$$\bar{d}(\psi_2, id) \leq d_{C^0}(\psi_2, id) + d_{C^0}(\psi_2^{-1}, id) \leq 2r(d_{\mathcal{M}}(m_f, m_g); g).$$

This finishes the proof of (1) of Theorem 3.2 and hence the proof of Theorem 3.2.

7. Proof of Theorem II

With Proposition 3.1 and Theorem 3.2 in our hand, we now give the proof of Theorem II. We will imitate Moser's argument [Mo] but with some additional arguments needed to establish the C^0-coerciveness.

Let $\tau = \Omega$ be a volume form on X, $f > 0$ be a positive function on X and $\sigma = f\Omega$. We choose an open covering $\mathcal{U} = \{U_j\}$ of X and denote by $m = m_{\mathcal{U}}$ the cardinality of $\mathcal{U}$.

Consider the functions $f(t;\cdot)$ defined by

$$f(t;p) = 1 + \sum_{j=0}^{m} t_j g_j(p), \quad t = (t_0, \cdots, t_m).$$

For $t = (0, \cdots, 0)$, one has $f(t;p) \equiv 0$ and for $t = (1, \cdots, 1)$, we have $f(t;\cdot) = f$. By construction, we also have $\int f(t;\cdot)\Omega = \int \Omega$ for all t. We can connect two corners $(0, \cdots, 0)$ and $(1, \cdots, 1)$ of the cube by going along $m+1$ edges. If t', t'' represent the endpoints of such an edge, one sees that

$$f(t'';\cdot) - f(t';\cdot)$$

has support in one patch, say $U_{\{t',t''\}}$. Without loss of any generality, we may parameterize

$$U_{\{t',t''\}} \cong (-\eta, 1+\eta)^n$$

for some $\eta > 0$ and

$$\mathrm{supp}(f(t'';\cdot) - f(t';\cdot)) \subset Q^n(1-\eta)$$

under the parametrization. If we write

$$\Omega_t = f(t;\cdot)\Omega$$

one sees that $\Omega_{t''} = h\Omega_{t'}$ where $h = f(t'';\cdot)/f(t';\cdot)$ is different from 1 in $U_{\{t',t''\}}$ only and $h \equiv 1$ on $Q^n(1+\eta) \setminus Q^n(1-\eta)$.

Once we have made the choice of such a covering U_j, we consider the family

$$(t,s) \in [0,1]^{n+1} \times [0,1] \mapsto f(t,s)$$

where $f(t,s)$ is the function on X defined by

$$f(t,s;p) = 1 + s\sum_{j=0}^{m} t_j g_j(p). \tag{7.1}$$

We partition $s \in [0,1]$ into a partition

$$P : s_0 = 0 < s_1 < \cdots < s_N.$$

By choosing P with $\mathrm{mesh}(P)$ sufficiently small, we can make

$$d_{\mathcal{M}}(m_{f(t',s_i)}, m_{f(t'',s_i)}) \tag{7.2}$$

as small as we want uniformly over (t', t''). Therefore we will assume that $d_{\mathcal{M}}(\mu_\Omega, \mu_{f\Omega})$ is so small that we can apply Proposition 6.3.

We order the set U_j so that $U_j = U_{\{t_j,t_{j+1}\}}$ where $U_{\{t_j,t_{j+1}\}}$ is the patch corresponding to $(t', t'') = (t_j, t_{j+1})$. Now applying Theorem 3.2 to each patch U_j, we have constructed a sequence of diffeomorphisms $\phi_j : X \to X$ such that

(1) $\phi_{j+1} \circ \phi_j^{-1}$ has support in $Q^n(1-\eta) \subset U_j \cong Q^n(1+\eta)$

(2) $\Omega_{t_{j+1}} = \phi_j^*\Omega_{t_j}$ or equivalently $(\phi_i)_*\mu_{j+1} = \mu_j$ where $\mu_j = \mu_{\Omega_{t_j}}$. Here we denote by t_j the j-th vertex in the above chosen edge path from $(0,\cdots,0)$ to $(1,\cdots,1)$.

Then the diffeomorphism $\psi_2 = \phi_m : X \to X$ satisfies $\psi_2^*\Omega = f\Omega$.

It remains to estimate $\bar{d}(\psi_2, id)$. Since $d_{C^0}(\psi_2^{-1}, id) = d_{C^0}(id, \psi_2)$, it is enough to estimate $d_{C^0}(\psi_2, id)$. Denote the above coordinate patch map

$$\psi_j : U_j \to (-\eta, 1+\eta)^n.$$

Then the above diffeomorphism $\phi_{j+1}\circ\phi_j^{-1}$ is given by the conjugation

$$\phi_{j+1}\circ\phi_j^{-1} = \psi_j^{-1}\circ\varphi_j\circ\psi_j$$

where $\varphi_j : Q^n(1+\eta) \to Q^n(1+\eta)$ is the diffeomorphism constructed in section 6 corresponding to the forms

$$(f_{j+1}\circ\psi_{j+1}^{-1})\,dx, \quad (f_j\circ\psi_j^{-1})\,dx.$$

Because we will use the C^0-norm in different spaces, we will specify the space where the C^0-norm is taken below when we need to specify the space. We have from Proposition 6.3

$$d_{(C^0,Q^n)}(\varphi_j, id) \le C_5(d_{\mathcal{M}}(m_{j+1}, m_j); f_j\circ\psi_j^{-1}) \tag{7.3}$$

where m_j is the measure associated to the form $(f_j\circ\psi_j^{-1})\,dx$. We recall that the finite family of functions $f_j\circ\psi_j^{-1}$ are determined by the original function f, the covering $\mathcal{U}$ and the coordinate charts ψ_j. Since we do not change but fix them in the course of proof, we may ignore this dependence of C_5 on $f_j\circ\psi_j^{-1}$.

We note that

$$d_{(C^0,X)}(\phi_{j+1}\circ\phi_j, id) = d_{C^0}(\psi_j^{-1}\circ\varphi_j\circ\psi_j, id) \to 0$$

as $d_{(C^0,Q^n(1+\eta))}(\varphi_j, id) \to 0$ and

$$d_{(C^0,X)}(\psi_2, id) \le \sum_{j=0}^{m-1} d_{(C^0,X)}(\phi_{j+1}, \phi_j). \tag{7.4}$$

On the other hand, we have

$$\begin{aligned} d_{(C^0,X)}(\phi_{j+1}, \phi_j) &= d_{(C^0,X)}((\phi_{j+1}\circ\phi_j^{-1})\circ\phi_j, \phi_j) \\ &= d_{(C^0,X)}(\phi_{j+1}\circ\phi_j^{-1}), id) \\ &= d_{(C^0,X)}(\psi_j^{-1}\circ\varphi_j\circ\psi_j, id). \end{aligned} \tag{7.5}$$

Since the integer $m = m_{\mathcal{U}} < \infty$ depends only on $\mathcal{U}$ but not on f or j's, it follows that as $d_{\mathcal{M}}(\mu_{f\Omega}, \mu_\Omega) \to 0$, $\bar{d}(\psi_2, id) \to 0$. This finishes the proof of Theorem II except its parameterized version.

For the parameterized version, we recall that $\tau = \Omega$ is fixed and that $\sigma_s = f_s\Omega$ for a given a smooth family of functions f_s for $s \in [0,1]$, for which $s \mapsto \mu_{(f_s\Omega)}$ is continuous in $\mathcal{M}(X)$. Note that the above mentioned covering $\mathcal{U} = \{U_j\}$ in section 3 does not depend on the functions f_s and so can be fixed for all $s \in [0,1]$. This and the compactness of $[0,1]$ enable us to reduce the problem to the parameterized version of Theorem 3.2 on the cube for a fixed g but varying f_s in a way that $s \mapsto \mu_{(f_s dx)} = m_{f_s}$ is continuous in $\mathcal{M}(Q^n)$. Since all the constants appearing in section 6 depend continuously on $d_{\mathcal{M}}(m_f, m_g)$, we can uniformly apply Dacorogna-Moser's construction to produce an isotopy $s \mapsto \psi_{2,s}$ of diffeomorphisms that is

continuous in compact open topology of $Diff(X)$ and satisfies $(\psi_{2,s}^{-1})^*\Omega = f_s\Omega$ which is equivalent to $\Omega = \psi_{2,s}^*(f_s\Omega)$.

One particular remark on the choice of the constant $\varepsilon_0 = \varepsilon_0(g,\zeta)$ in our construction on the cube Q^n is in order for the parameterized case. For the given isotopy $\mathcal{F} = \{f_s\}_{0\leq s\leq 1}$ on M, we can reduce the problem to the cube so that

$$\mathrm{supp}(f_s - g) \subset Q^n(1-\eta)$$

for all $s \in [0,1]$. (See section 3.) Then we choose $\varepsilon_0^{\mathcal{F}} = \varepsilon_0(\mathcal{F},\zeta)$ by

$$\varepsilon_0^{\mathcal{F}} = \min_{s\in[0,1]} \varepsilon_0(f_s,\zeta)$$

which can be made close to 0 uniformly over s by choosing the family $\zeta = \{\zeta_s\}_{s=2}^n$ of cut-off functions ζ as in (4.22) suitably.

To improve the regularity of the parameterized solutions, we use the a priori C^k estimate provided in Proposition 4.1. This finishes the proof of Theorem II

8. Proof of Theorem I′

In this section, we finish the proof of Theorem I′ following the scheme outlined in the introduction.

Let $h \in M[\Sigma,\Omega]$ and $\varepsilon > 0$ be given. By the smoothing theorem (see Theorem 6.3 [Mu2] for example), we can choose a diffeomorphism ψ_1 such that

$$\bar{d}(h,\psi_1) \leq \frac{\varepsilon}{2}. \tag{8.1}$$

This diffeomorphism is *not* necessarily area preserving. We therefore modify ψ_1 into an area preserving diffeomorphism by composing it with another diffeomorphism $\psi_2 : \Sigma \to \Sigma$ that is C^0-close to the identity.

It follows from Proposition 2.1 that (8.1) also implies that the measures associated to $\mu_\Omega = h_*(\mu_\Omega)$ and $(\psi_1^{-1})_*(\mu_\Omega) = \mu_{((\psi_1)^*\Omega)}$ can be made arbitrarily close in the weak topology of measures.

We note that $\int_\Sigma(\psi_1)^*\Omega = \int_\Sigma \Omega$. Therefore applying Theorem II to the forms

$$\sigma = \Omega, \quad \tau = (\psi_1^{-1})^*\Omega, \quad \text{with } \lambda = 1$$

we obtain a diffeomorphism ψ_2 such that

$$\psi_2^*\Omega = (\psi_1^{-1})^*\Omega$$

and

$$\bar{d}(\psi_2, id) \to 0 \quad \text{as } d_{\mathcal{M}}(\mu_{((\psi_1)^*\Omega)}, \mu_\Omega) \to 0. \tag{8.2}$$

We set $\phi = \psi_2 \circ \psi_1$. Then ϕ is an μ_Ω-area preserving diffeomorphism and we have

$$\bar{d}(\phi,h) \leq \bar{d}(\psi_2\circ\psi_1,\psi_1) + \bar{d}(\psi_1,h) \tag{8.3}$$

by the triangle inequality. But since X is compact and ψ_1 is a diffeomorphism, we can make $\bar{d}(\psi_2\circ\psi_1,\psi_1)$ as small as we want by having $d(\psi_2,id)$ sufficiently small. However (8.2) implies that $d(\psi_2,id)$ can be made as small as we want if we can let $d_{\mathcal{M}}(\mu_{(\psi_1^*\Omega)},\mu_\Omega) = d_{\mathcal{M}}((\psi_1^{-1})_*\mu_\Omega,\mu_\Omega)$ arbitrarily small. And the latter can be achieved by Proposition 2.1 if we choose the initial diffeomorphism ψ_1 sufficiently C^0 close to h. Combining these with (8.3), we can make

$$\bar{d}(\phi,h) \leq \bar{d}(\psi_2\circ\psi_1,\psi_1) + d(\psi_1,h) \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

if we choose the initial smooth approximation ψ_1 sufficiently C^0 close to h. This finishes the proof of Theorem I′ (1).

Finally when we are given an isotopy $\{h_t\}_{0\leq t\leq 1}$ of homeomorphisms $h_t : \Sigma \to \Sigma$, we apply the isotopy version of smoothing theorem (see Theorem 6.3 [Mu2]) to obtain an isotopy of diffeomorphisms $\{\psi_{1,t}\}_{0\leq t\leq 1}$ so that $\bar{d}(\psi_{1,t}, h_t)$ can be made as small as we want uniformly over $0 \leq t \leq 1$.

Then the isotopy of forms

$$t \in [0,1] \mapsto (\psi_{1,t})^*\Omega$$

is continuous in the sense mentioned in Theorem II. Therefore we can apply the parameterized version of Theorem II to produce another isotopy $\{\psi_{2,t}\}_{0\leq t\leq 1}$ so that

(1) $\psi_{1,t}^*(\psi_{2,t}^*\Omega) = \Omega$ for all $0 \leq t \leq 1$.
(2) The isotopy $t \mapsto \psi_{2,t}$ is continuous in compact open topology.
(3) $\bar{d}(\psi_{2,t}, id)$ is as small as we want uniformly over $t \in [0,1]$.

Now the composed isotopy defined by $\psi_t = \psi_{2,t} \circ \psi_{1,t}$ will do our purpose. This finishes the proof of Theorem I′.

Remark 8.1. Here we would like to point out that we should apply our construction to the *fixed* form Ω, not to the *varying* form $\psi_1^*\Omega$. In this way, the dependence on g appearing in all the constants in section 6 become irrelevant in our construction because this function g will be fixed throughout the construction. This is the reason why we consider the pair of forms

$$\sigma = \Omega, \quad \tau = (\psi_1^{-1})^*\Omega$$

instead of the more naturally looking choice of

$$\sigma = (\psi_1)^*\Omega, \quad \tau = \Omega.$$

9. Estimates of the higher order terms

In this section, we prove Lemma 6.2. We would like to note that the nonlinear terms depends only on g. Dependence on f occurs only through the constants $\varepsilon_0(\zeta)$ and $\varepsilon_1(\zeta)$ which involves the choice of cut-off functions ζ. However these constants can be made as small as we want, e.g., smaller than $d_{\mathcal{M}}(m_f, m_g)$ as in (4.23) and (6.9) by choosing ζ appropriately, once f is given.

We start with the statement (1) of Lemma 5.2. We recall from (6.4)

$$\begin{aligned} N_j(u) \quad &= \quad \int_{R^n_{au;j}} g(y)dy - \int_{Q^n_{a;j}} g(y)dy \\ &\qquad - \sum_{k=j}^{n} \int_{Q^{n-k}_{a;j}} u_k(a_k, \widetilde{x}_k) \Big(\int_{Q^{k-1}_{a;j,k}} \zeta_j(x^{k-1}) g(x^{n-1}_{a;k}) dx^{k-1} \Big) d\widetilde{x}_k. \end{aligned}$$

Now to estimate (6.4), we rewrite $v(Q^n_{a;j})$ as

$$v(Q^n_{a;j}) = \varphi_n \circ \varphi_{n-1} \circ \cdots \circ \varphi_j(Q^n_{a;j})$$

by the factorization of $v = \varphi_n \circ \varphi_{n-1} \circ \cdots \circ \varphi_1$. Here we also use the identity

$$\varphi_{j-1} \circ \cdots \circ \varphi_1(Q^n_{a;j}) = Q^n_{a;j}$$

which follows from the definitions of φ_k and $Q^n_{a;j}$. Motivated by this, for each $j-1 \leq l \leq n+1$, we define

$$R^n_{au,l;j} = \varphi_n \circ \cdots \circ \varphi_l(Q^n_{a;j}).$$

We note that $R^n_{au,j;j} = R^n_{au;j}$ and set $R^n_{au,n+1;j} = Q^n_{a;j}$.

From now on, we will switch the variable y with x and use x instead of y for the rest of the proof.

With these definitions, we can telescope and rewrite

$$\int_{R^n_{au;j}} g(x)dx - \int_{Q^n_{a;j}} g(x)dx = \sum_{l=j}^{n} \left(\int_{R^n_{au,l;j}} g(x)dx - \int_{R^n_{au,l+1;j}} g(x)dx \right).$$

On the other hand, we can easily check

$$\begin{aligned} &\int_{Q^{n-k}_{a;j}} u_k(a_k, \widetilde{x}_k)\Big(\int_{Q^{k-1}_{a;j,k}} \zeta_j(x^{k-1}) g(x^{n-1}_{a;k}) dx^{k-1}\Big) d\widetilde{x}_k = \\ &\int_{Q^n_{a;j,k} \setminus Q^n_{a;j}} \zeta_k(x^{k-1}) g(x^{n-1}_{a;k}) dx - \int_{Q^n_{a;j} \setminus Q^n_{a;j,k}} \zeta_k(x^{k-1}) g(x^{n-1}_{a;k}) dx \end{aligned} \tag{9.1}$$

where $Q^n_{a;j,k}$ is defined by

$$\begin{aligned} Q^n_{a;j,k} = \{x \in Q^n \quad | \quad & 0 \leq x_i \leq 1 \text{ for } i = 1, \cdots, j-1, \\ & 0 \leq x_k \leq a_k + u_k(a_k, \widetilde{x}_k) \\ & 0 \leq x_i \leq a_i \text{ for } i \geq j \,\&\, i \neq k\}. \end{aligned}$$

Here we would like to note that the sign of u_k could be either positive or negative. Then we have

$$\begin{aligned} N_j(u) &= \sum_{l=0}^{n-j} \left(\int_{R^n_{au,j+l;j}} g(x)dx - \int_{R^n_{au,j+l+1;j}} g(x)dx \right) \\ &\quad - \sum_{l=0}^{n-j} \Big(\int_{Q^n_{a;j,j+l} \setminus Q^n_{a;j}} \zeta_{j+l}(x^{j+l-1}) g(x^{n-1}_{a;j+\ell}) dx \\ &\qquad\qquad - \int_{Q^n_{a;j} \setminus Q^n_{a;j,j+l}} \zeta_{j+l}(x^{j+l-1}) g(x^{n-1}_{a;j+\ell}) dx \Big) \\ &= \sum_{l=0}^{n-j} \left(\int_{R^n_{au,j+l;j} \setminus R^n_{au,j+l+1;j}} g(x)dx - \int_{R^n_{au,j+l+1;j} \setminus R^n_{au,j+l;j}} g(x)dx \right) \\ &\quad - \sum_{l=0}^{n-j} \Big(\int_{Q^n_{a;j,j+l} \setminus Q^n_{a;j}} \zeta_{j+l}(x^{j+l-1}) g(x^{n-1}_{a;j+\ell}) dx \\ &\qquad\qquad - \int_{Q^n_{a;j} \setminus Q^n_{a;j,j+l}} \zeta_{j+l}(x^{j+l-1}) g(x^{n-1}_{a;j+\ell}) dx \Big). \end{aligned}$$

For the simplicity of notations, we will just denote

$$\zeta_{j+l} = \zeta_{j+l}(x^{j+l-1})$$

for the rest of the paper. We now estimate

$$\left(\int_{R^n_{au,j+l;j}\setminus R^n_{au,j+l+1;j}} g(x)dx - \int_{R^n_{au,j+l+1;j}\setminus R^n_{au,j+l;j}} g(x)dx\right)$$
$$-\left(\int_{Q^n_{a;j,j+l}\setminus Q^n_{a;j}} \zeta_{j+l}\cdot g(x^{n-1}_{a;j+\ell})dx - \int_{Q^n_{a;j}\setminus Q^n_{a;j,j+l}} \zeta_{j+l}\cdot g(x^{n-1}_{a;j+\ell})dx\right)$$

for each $l = 0, 1, \cdots, n-j$. We further rewrite it as

$$\begin{aligned}&\left(\int_{R^n_{au,j+l;j}\setminus R^n_{au,j+l+1;j}} g(x)dx - \int_{R^n_{au,j+l+1;j}\setminus R^n_{au,j+l;j}} g(x)dx\right.\\
&\quad - \int_{R^n_{au,j+l;j}\setminus R^n_{au,j+l+1;j}} \zeta_{j+l}\cdot g(x^{n-1}_{a;j+l})dx\\
&\qquad\left. - \int_{R^n_{au,j+l+1;j}\setminus R^n_{au,j+l;j}} \zeta_{j+l}\cdot g(x^{n-1}_{a;j+l})dx\right) \qquad (9.2)\\
&+\left(\int_{R^n_{au,j+l;j}\setminus R^n_{au,j+l+1;j}} \zeta_{j+l}\cdot g(x^{n-1}_{a;j+l})dx\right.\\
&\qquad - \int_{R^n_{au,j+l+1;j}\setminus R^n_{au,j+l;j}} \zeta_{j+l}\cdot g(x^{n-1}_{a;j+l})dx\\
&\quad\left. - \int_{Q^n_{a;j,j+l}\setminus Q^n_{a;j}} \zeta_{j+l}\cdot g(x^{n-1}_{a;j+l})dx - \int_{Q^n_{a;j}\setminus Q^n_{a;j,j+l}} \zeta_{j+l}\cdot g(x^{n-1}_{a;j+l})dx\right) \qquad (9.3)\end{aligned}$$

We estimate (9.2) and (9.3) separately. We start with (9.2).

The terms in (9.2) can be combined into

$$\begin{aligned}&\left(\int_{R^n_{au,j+l;j}\setminus R^n_{au,j+l+1;j}} g(x)dx - \int_{R^n_{au,j+l+1;j}\setminus R^n_{au,j+l;j}} g(x)dx\right)\\
&\quad -\left(\int_{R^n_{au,j+l;j}\setminus R^n_{au,j+l+1;j}} \zeta_{j+l}\cdot g(x^{n-1}_{a;j+l})dx\right.\\
&\qquad\left. - \int_{R^n_{au,j+l;j}\setminus R^n_{au,j+l;j}} \zeta_{j+l}\cdot g(x^{n-1}_{a;j+l})dx\right)\\
= &\int_{R^n_{au,j+l;j}\setminus R^n_{au,j+l+1;j}} \left(g(x) - \zeta_{j+l}\cdot g(x^{n-1}_{a;j+l})\right)dx\\
&\qquad - \int_{R^n_{au,j+l+1;j}\setminus R^n_{au,j+l;j}} \left(g(x) - \zeta_{j+l}\cdot g(x^{n-1}_{a;j+l})\right)dx \qquad (9.4)\end{aligned}$$

Then we obtain the inequality

$$m(R^n_{au,j+l;j}\setminus R^n_{au,j+l+1;j}) \le \left(\prod_{i=j}^{j+l} a_i\right)\cdot|\zeta_{j+l+1}u_{j+l+1}|\cdot\left(\prod_{i=j+2+l}^{n}|a_i+\zeta_i u_i(a)|\right) \le |\zeta\cdot u|$$

and the same for $m(R^n_{au,j+l+1;j}\setminus R^n_{au,j+l;j})$.

We have the bound for the first term of (9.4)

$$
\begin{aligned}
& \Big| \int_{R^n_{au,j+l;j} \setminus R^n_{au,j+l+1;j}} \Big(g(x) - \zeta_{j+l} \cdot g(x^{n-1}_{a;j+l}) \Big)\, dx \Big| \\
\leq \; & \Big| \int_{R^n_{au,j+l;j} \setminus R^n_{au,j+l+1;j}} \Big(g(x) - g(x^{n-1}_{a;j+l}) \Big)\, dx \Big| \qquad (9.5)\\
& + \int_{R^n_{au,j+l;j} \setminus R^n_{au,j+l+1;j}} |1 - \zeta_{j+l}| g(x^{n-1}_{a;j+l}) dx. \qquad (9.6)
\end{aligned}
$$

To get a bound for (9.5), we now define

$$
R^{n-1}_{au;j,k}(a) = \{(x_1, \cdots, \widehat{x}_k, x_{k+1}, \cdots, x_n) \mid (x_1, \cdots, a_k, x_{k+1}, \cdots, x_n) \in R^n_{au;j}\} \tag{9.7}
$$

for each $j \leq k \leq n$. Fubini's theorem then implies that the term (9.5) can be bounded by

$$
\begin{aligned}
& \int_{R^n_{au,j+l;j} \setminus R^n_{au,j+l+1;j}} |g(x) - g(x^{n-1}_{a;j+l})| dx \\
= & \int_{R^{n-1}_{a;j,j+l}(a)} \int_{M^-_{j+l+1}(a;u)}^{M^+_{j+l+1}(a;u)} |g(x) - g(x^{n-1}_{a;j+l})| dx
\end{aligned}
$$

where we define the constants

$$
M^-_k(a; \zeta \cdot u) = \min\{a_k, a_k + \zeta_k u_k(a)\}, \quad M^+_k(a; \zeta \cdot u) = \max\{a_k, a_k + \zeta_k u_k(a)\}.
$$

Noting that

$$
M^+_{j+l+1}(a; u) - M^-_{j+l+1}(a; u) = |\zeta_{j+l+1} u_{j+l+1}(a)|,
$$

we have

$$
|x - x^{n-1}_{a;j+l}| \leq |\zeta_{j+l+1} u_{j+l+1}(a)| \leq |\zeta \cdot u|
$$

for all

$$
x \in R^n_{au,j+l;j} \setminus R^n_{au,j+l+1;j} \quad \text{and} \quad \pi_{j+l+1}(x) = \pi_{j+l+1}(x^{n-1}_{a;j+l+1}) \tag{9.8}
$$

where $\pi_{j+l+1} : \mathbb{R}^n \to \mathbb{R}^{n-1}$ is the coordinate projection along the $(j+l+1)$-th axis.

Using this preparation, we have the bound for(9.5) given by

$$
|\zeta \cdot u| \cdot \max |g(x) - g(x^{n-1}_{a;j+l})| :
$$

Here the maximum is taken over all x, $x^{n-1}_{a;j+l+1}$ satisfying (9.8). *Using the continuity of g and compactness of Q*, we have

$$
|g(x) - g(y)| \leq L_g |x - y| \tag{9.9}
$$

for some $L_g > 0$ depending only on g. We also note

$$
|a_i + \zeta_i u_i(a)| \leq 1.
$$

Therefore (9.5) is bounded by

$$
|\zeta_{j+l} u_{j+l}| L_g |\zeta \cdot u| \leq 2 L_g |\zeta \cdot u| |\zeta \cdot u|. \tag{9.10}
$$

On the other hand, Fubini's theorem implies that the term (9.6) can be bounded by

$$\begin{aligned}
&\int_{R^n_{au,j+l;j}\setminus R^n_{au,j+l+1;j}} |1-\zeta_{j+l}|g(x^{n-1}_{a;j+l})dx \\
=& \int_{R^{n-1}_{a;j,j+l}(a)} \int_{M^-_{j+l+1}(a;u)}^{M^+_{j+l+1}(a;u)} |1-\zeta_{j+l}|g(x^{n-1}_{a;j+l})dx. \qquad (9.11)
\end{aligned}$$

Now we define $\varepsilon_1(\zeta)$ to be

$$\varepsilon_1(\zeta) = \max_{1\le k\le n}\left(\int_{Q^{k-1}} |1-\zeta_k(x^{k-1})|dx^{k-1}\right)$$

as in (6.9). With this definition and by Fubini's theorem, we estimate (9.11) by

$$\begin{aligned}
&\int_{R^{n-1}_{a;j,j+l}(a)} \int_{M^-_{j+l+1}(a;u)}^{M^+_{j+l+1}(a;u)} |1-\zeta_{j+l+1}|g(x^{n-1}_{a;j+l+1})dx \\
\le\ & \max g \int_{Q^{j+l}} \int_{Q^{n-(j+l+1)}_{a;j+l}} \int_{M^-_{j+l+1}(a;u)}^{M^+_{j+l+1}(a;u)} |1-\zeta_{j+l+1}(x^{j+l})|dx \\
\le\ & \max g|\zeta\cdot u| \int_{Q^{j+l}_{a;j+l}} |1-\zeta_{j+l+1}|dx \\
\le\ & \varepsilon_1(\zeta)|\zeta\cdot u|\max g. \qquad (9.12)
\end{aligned}$$

Here we denote

$$Q^{n-k}_{a;k} = \{(a_k,\widetilde{x}_k) \mid \widetilde{x}_k \in Q^{n-(k+1)}\}.$$

Since g is fixed, we can make $\varepsilon_1(\zeta)$ as small as we want by choosing the cut-off function ζ as close to 1 as possible in the L^1 sense.

Applying the above discussion of (9.5) and (9.6), we derive

$$\Big|\int_{R^n_{au,j+l;j}\setminus R^n_{au,j+l+1;j}} \Big(g(x)-\zeta_{j+l+1}\cdot g(x^{n-1}_{a;j+l+1})\Big)\,dx\Big| \le (\varepsilon_1(\zeta)\max g + 2L_g|\zeta\cdot u|)|\zeta\cdot u|$$

from (9.2) and (9.12). Here we use the assumption that $|\zeta\cdot u|\le 2$ and the fact $a_i<1$. Similar estimate gives the bound

$$\Big|\int_{R^n_{au,j+l+1;j}\setminus R^n_{au,j+l;j}} \Big(g(x)-\zeta_{j+l+1}\cdot g(x^{n-1}_{a;j+l+1})\Big)\,dx\Big| \le (\varepsilon_1(\zeta)\max g + 2L_g|\zeta\cdot u|)|\zeta\cdot u| \qquad (9.13)$$

and hence (9.2) is bounded by

$$2(\varepsilon_1(\zeta)\max g + 2L_g|\zeta\cdot u|)|\zeta\cdot u|. \qquad (9.14)$$

Next, we turn to (9.3). The four terms in (9.3) can be combined into

$$\begin{aligned}
&\Bigg(\int_{R^n_{au,j+l;j}\setminus R^n_{au,j+l+1;j}} \zeta_{j+l}\cdot g(x^{n-1}_{a;j+l})dx - \int_{R^n_{au,j+l+1;j}\setminus R^n_{au,j+l;j}} \zeta_{j+l}\cdot g(x^{n-1}_{a;j+l})dx \\
&\quad - \int_{Q^n_{a;j,j+l}\setminus Q^n_{a;j}} \zeta_{j+l}\cdot g(x^{n-1}_{a;j+l})dx - \int_{Q^n_{a;j}\setminus Q^n_{a;j,j+l}} \zeta_{j+l}\cdot g(x^{n-1}_{a;j+l})dx\Bigg) \\
&= \Bigg(\int_{R^n_{au,j+l;j}\setminus R^n_{au,j+l+1;j}} \zeta_{j+l}\cdot g(x^{n-1}_{a;j+l})dx - \int_{Q^n_{a;j,j+l}\setminus Q^n_{a;j}} \zeta_{j+l}\cdot g(x^{n-1}_{a;j+l})dx\Bigg) \\
&\quad - \Bigg(\int_{R^n_{au,j+l+1;j}\setminus R^n_{au,j+l;j}} \zeta_{j+l}\cdot g(x^{n-1}_{a;j+l})dx - \int_{Q^n_{a;j}\setminus Q^n_{a;j,j+l}} \zeta_{j+l}\cdot g(x^{n-1}_{a;j+l})dx\Bigg)
\end{aligned}$$

Here we can write

$$\begin{aligned}
& \int_{R^n_{au,j+l;j}\setminus R^n_{au,j+l+1;j}} \zeta_{j+l}\cdot g(x^{n-1}_{a;j+l})dx - \int_{Q^n_{a;j,j+l}\setminus Q^n_{a;j}} \zeta_{j+l}\cdot g(x^{n-1}_{a;j+l})dx \\
= & \int_{\gamma_{j+l+1}(\varphi_{j+l}(Q^n_{a;j})\setminus Q^n_{a;j}))} \zeta_{j+l}\cdot g(x^{n-1}_{a;j+l})dx - \int_{Q^n_{a;j,j+l}\setminus Q^n_{a;j}} \zeta_{j+l}\cdot g(x^{n-1}_{a;j+l})dx \\
= & \int_{(\gamma_{j+l+1}(\varphi_{j+l}(Q^n_{a;j})\setminus Q^n_{a;j}))\setminus(Q^n_{a;j,j+l}\setminus Q^n_{a;j})} \zeta_{j+l}\cdot g(x^{n-1}_{a;j+l})dx \\
& \quad - \int_{(Q^n_{a;j,j+l}\setminus Q^n_{a;j})\setminus(\gamma_{j+l+1}(\varphi_{j+l}(Q^n_{a;j})\setminus Q^n_{a;j}))} \zeta_{j+l}\cdot g(x^{n-1}_{a;j+l})dx
\end{aligned}$$

where $\gamma_{j+l+1}: Q^n \to Q^n$ is the diffeomorphism

$$\gamma_{j+l+1} = \varphi_n \circ \cdots \circ \varphi_{j+l+1}.$$

Now we prove the following lemma

Lemma 9.1. *Let* $x \in (\gamma_{j+l+1}(\varphi_{j+l}(Q^n_{a;j,j+l}) \setminus Q^n_{a;j})) \setminus (Q^n_{a;j,j+l} \setminus Q^n_{a;j})$. *For each* $0 \le l < n-j$, *x must satisfy*

$$\begin{array}{rcll}
0 & \le & x_i \le 1 & \textit{for } 1 \le i \le j-1 \\
0 & \le & x_i \le a_i & \textit{for } j \le i \le j+l-1 \\
M_k^-(a;\zeta\cdot u) & \le & x_k \le M_k^+(a;\zeta\cdot u) & \textit{for some } j+l+1 < k \le n \\
a_{j+l} + u_{j+l}(a_{j+l}, \widetilde{x}_{j+l}) & \le & x_{j+l} \le a_{j+l} + \zeta_{j+l} u_{j+l}(a_{j+l}, \widetilde{x}_{j+l}). &
\end{array}$$

For $l = n-j$, *we have*

$$(\gamma_{j+l+1}(\varphi_{j+l}(Q^n_{a;j,j+l})\setminus Q^n_{a;j}))\setminus(Q^n_{a;j,j+l}\setminus Q^n_{a;j}) = \varphi_n(Q^n_{a;j,j+l})\setminus Q^n_{a;j}))\setminus(Q^n_{a;j,j+l}\setminus Q^n_{a;j})$$

and

$$\begin{array}{rcll}
0 & \le & x_i \le 1 & \textit{for } 1 \le i \le n-1 \\
a_n + u_n(a_n) & \le & x_n \le a_n + \zeta_n(x^{j+l-1})u_n(a_n). &
\end{array}$$

Proof. First consider the case $0 \le l < n-j$. Note that

$$\begin{aligned}
\gamma_{j+l+1}(x_1, \cdots, x_n) \quad = \quad & (x_1, \cdots, x_{j+l}, x_{j+l+1} + \zeta_{j+l+1}(x^{j+l})u_{j+l+1}(x_{j+l+1}, \widetilde{x}^{j+l+1}), \\
& \cdots x_n + \zeta_n(\varphi_{n-1} \circ \cdots \circ \varphi_{j+l}(x))u_n(x_n)).
\end{aligned}$$

Let

$$\begin{aligned} x \quad \in \quad & (\gamma_{j+l+1}(\varphi_{j+l}(Q^n_{a;j,j+l}) \setminus Q^n_{a;j})) \setminus (Q^n_{a;j,j+l} \setminus Q^n_{a;j}) \\ & = \gamma_{j+l+1}(\varphi_{j+l}(Q^n_{a;j,j+l})) \setminus (Q^n_{a;j,j+l} \cup Q^n_{a;j}). \end{aligned}$$

Then we first have

$$a_{j+l} \le x_{j+l} \le a_{j+l} + \zeta_{j+l} u_{j+l}(a_{j+l}, \widetilde{x}_{j+l}).$$

(Here if $u_{j+l}(a_{j+l}, \widetilde{x}_{j+l}) < 0$, this inequality is vacuous, i.e., no such x exists.)

And for some $k \ge j+l+1$, we have $x_k \ge a_k$ and can write

$$x_k = y_k + \zeta_k u_k(y_k, \widetilde{y}_k)$$

for some y satisfying

$$y_k \le a_k.$$

In particular, we obtain

$$a_k \le x_k \le a_k + \zeta_k u_k(a_k, \widetilde{x}_k)$$

and hence we have obtained

$$(M^-_k(a; \zeta \cdot u) \le) a_k \le x_k \le M^+_k(a; \zeta \cdot u).$$

The proof of other inequalities are easy and so omitted.

For the case $l = n - j$, we just note

$$\gamma_{n+1} = id$$

and then the rest follows. □

Noting $0 \le a_i \le 1$, $|\zeta_{j+l}| \le 1 + \varepsilon \le 2$ and

$$M^+_k(a; \zeta \cdot u) - M^-_k(a; \zeta \cdot u) = |\zeta_k u_k(a)|$$

we obtain

$$\begin{aligned} & m\left((R^n_{au,j+l;j} \setminus R^n_{au,j+l+1;j}) \setminus (Q^n_{a;j,j+l} \setminus Q^n_{a;j})\right) \\ \le \quad & \left(\prod_{j \le i < j+l} |a_i|\right) \cdot |\zeta_{j+l+1} u_{j+l+1}| |(1 + \zeta_{j+l+1}) u_{j+l+1}| \\ \le \quad & (2+\varepsilon) |\zeta \cdot u| |u| \end{aligned} \tag{9.15}$$

for $j \le l < n - j$. Therefore for $j \le l < n - j$, we have proved

$$m\left((R^n_{au,j+l;j} \setminus R^n_{au,j+l+1;j}) \setminus (Q^n_{a;j,j+l} \setminus Q^n_{a;j})\right) \le 3|\zeta \cdot u||u|. \tag{9.16}$$

Similarly we prove

$$m\left((Q^n_{a;j,j+l} \setminus Q^n_{a;j}) \setminus (R^n_{au,j+l;j} \setminus R^n_{au,j+l+1;j})\right) \le 3|\zeta \cdot u||u|. \tag{9.17}$$

Then we have proved that the absolute value of (9.3) is less than or equal to

$$6(\max g)|\zeta \cdot u||u| \tag{9.18}$$

for $j \le l < n - j$. (Here we need to treat the case of $j = 1$ slightly differently but again the same inequality can be shown to hold whose details we leave for the readers.)

On the other hand, when $l = n - j$, (9.3) can be estimated as

$$\begin{aligned}
&\Big| \int_{(\varphi_n(Q^n_{a;j})\setminus Q^n_{a;j})\setminus(Q^n_{a;j,n}\setminus Q^n_{a;j})} \zeta_{j+l}\cdot g(x^{n-1}_{a;j+l})dx \\
&\quad - \int_{(Q^n_{a;j,n}\setminus Q^n_{a;j})\setminus(\varphi_n(Q^n_{a;j})\setminus Q^n_{a;j})} \zeta_{j+l}\cdot g(x^{n-1}_{a;j+l})dx \Big| \\
\leq\ & 2\int_{Q^{n-1}}\int_{M_n^-(a;\zeta\cdot u)}^{M_n^+(a;\zeta\cdot u)} \zeta_n(x^{n-1}) g(x^{n-1}_{a;n})\, dx_n dx^{n-1} \\
\leq\ & 4|\zeta_n u_n(a_n)| \int_{Q^{n-1}} |1-\zeta(x^{n-1})| g(x^{n-1}_{a;n})\, dx^{n-1} \\
\leq\ & 4\varepsilon_1(\zeta)\max g|\zeta\cdot u|. \qquad (9.19)
\end{aligned}$$

Therefore combining (9.18) and (9.19), we have proved that (9.3) is bounded by

$$6(\max g)|\zeta\cdot u||u| + 4\varepsilon_1(\zeta)\max g|\zeta\cdot u| \qquad (9.20)$$

for any $1 \leq j \leq n$.

Combining (9.20) and (9.14), we have finally obtained

$$|N(u)| \leq (8\max g|\zeta\cdot u| + 6\max g\varepsilon_1(\zeta) + 4L_4|\zeta\cdot u|)\,|u|.$$

This finishes the proof of Lemma 6.2 by setting $C_4 = \max\{8\max g, 4\}$.

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